A Factorization Criterion for Route Invariants with Fixed Endpoint Data

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Abstract

A factorization criterion is proved for comparison data with fixed endpoints. A comparison framework consists of endpoints, routes, endpoint data, and route invariants. The endpoint map sends each route to the data determined by its source and target. The main result is that a route invariant is determined by endpoint data if and only if it is constant on every endpoint fiber, equivalently if and only if it factors through the endpoint quotient. Equivalently, endpoint closure holds precisely for the invariants that descend to the endpoint quotient. Endpoint data specify source, target, and any datum assigned to an ordered pair; they do not include the route itself. Thus any invariant separating routes with the same source, target, and endpoint datum is not endpoint-determined. With composition, a basic obstruction is the direct route versus a composite route. When a return route is available, the discrepancy can be completed to a loop. In smooth transport geometry, infinitesimal transport around an infinitesimal loop is represented by curvature. The proof uses elementary quotient descent for the endpoint map. Subsequent sections record instances in quotient semantics, categories, networks, gauge transport, Wilson loops, holonomy, differential geometry, and general relativity. The result is a descent criterion, not a derivation of those theories. The formal obstruction is that representative-, morphism-, path-, transport-, and loop-dependent information is not endpoint-determined unless it is constant on endpoint fibers.

Keywords. endpoint data; route invariants; quotient descent; holonomy; curvature; comparison data

Conventions. All collections below may be read as sets, or as classes in a fixed universe. When quotient notation is used, it denotes the ordinary set- or class-quotient by the displayed equivalence relation. No topological or smooth structure is assumed unless explicitly stated. The letters p, q, r denote endpoints; γ, η denote routes or paths; s, t denote source and target maps; C denotes endpoint-comparison data; e denotes the endpoint map; and π_E denotes the endpoint quotient map.

1 Endpoint Data and Route Invariants

The basic operation considered here is passage from routes to the data determined by their endpoints. This operation is a map from a finer description to coarser data, analogous to a quotient map or a projection to boundary data. A quantity descends along such a map exactly when it is constant on its fibers. In the present setting, those fibers consist of routes with the same endpoint data.

Write an endpoint comparison as

$$C(p, q),$$

where p and q are endpoints. The symbol may denote equality, equivalence, distance, causal relation, boundary value, quotient class, or any other datum assigned to the ordered pair. The datum is determined by endpoints. It contains neither a route between the endpoints, nor a composition law for comparisons, nor equality of route-dependent values for two routes from p to q.

The issue is factorization through the endpoint map, equivalently descent of the chosen invariant along endpoint projection. Endpoint data determine exactly the invariants that are constant on endpoint fibers. If an invariant distinguishes routes with the same source and target, then it is not determined by endpoint data. Representation of such an invariant requires more than endpoints.

Let $\mathcal{R}$ be a class of routes and let $\pi_E : \mathcal{R} \to \mathcal{R}/{\sim_E}$ send a route to its endpoint-data class. A route invariant I is determined by endpoint data if and only if it factors through π_E. Equivalently, I must be constant on every fiber of π_E. This is the ordinary descent condition for a quotient (Munkres, 2000).

Several standard settings instantiate the same factorization condition. For quotients, descent is constancy on equivalence classes (Munkres, 2000). In category theory, object data do not determine morphisms (Eilenberg and Mac Lane, 1945; Mac Lane, 1998). In networks, path products need not be determined by endpoints. In gauge theory, Wilson loops are gauge-invariant loop data rather than endpoint data (Wilson, 1974). In differential geometry, connections, holonomy, and curvature encode transport around loops (Ehresmann, 1951; Ambrose and Singer, 1953; Kobayashi and Nomizu, 1963; Nakahara, 2003). In general relativity, the corresponding hierarchy is metric comparison, Levi-Civita transport, curvature, and contraction, with curvature downstream of transport (Einstein, 1916; Wald, 1984; Malament, 2012; Carroll, 2019).

The sequence considered is

$$\boxed{\text{Endpoint} \longrightarrow \text{Triangle} \longrightarrow \text{Loop} \longrightarrow \text{Curvature.}}$$

The endpoint map defines the quotient. With composition, a basic comparison is the direct route $p \to r$ versus the composite comparison $p \to q \to r$. When a return route is available, completing that comparison produces transport around a loop. In smooth transport geometry, infinitesimal transport around an infinitesimal loop is represented by curvature.

Quotient topology, category theory, gauge theory, differential geometry, and general relativity have different structures. The claim here is not that these theories are derived from endpoint data. It is only that, in each setting, information depending on a representative, morphism, path, transport, or loop is determined by endpoint data only when it is constant on the relevant endpoint fibers.

2 Endpoint Fibers and Factorization

A map from finer descriptions to coarser data may be a quotient, an identification, a projection to boundary data, or the passage from routes to endpoint data. The quantities that descend along the map are exactly the quantities constant on its fibers.

Let X be a space of descriptions and let $q : X \to Q$ be such a map. A quantity $F : X \to S$ descends to Q precisely when there exists $\bar{F} : Q \to S$ such that

$$F = \bar{F} \circ q.$$

Equivalently, F is constant on the fibers of q. This is the ordinary quotient descent criterion (Munkres, 2000). The proof below is a structured use of that fact.

For comparison, the relevant map is the endpoint map. Routes may contain more information than endpoints. Endpoint comparison consists only of the endpoint class. The relevant obstruction is the endpoint fiber: the class of all routes with identical endpoint data.

Definition 2.1 (Endpoint comparison data)**.** *Endpoint comparison data consist of a class of endpoints $\mathcal{P}$ and, for each ordered pair (p, q) for which comparison is allowed, a datum $C(p, q)$. A route from p to q is an additional object denoted $\gamma : p \to q$. Two routes have the same endpoint data when they have the same source, the same target, and the same endpoint comparison datum.*

Definition 2.2 (Comparison framework with endpoint data)**.** *A comparison framework with endpoint data is a tuple*

$$\mathfrak{C} = (\mathcal{P}, \mathcal{R}, s, t, A, C, \mathsf{D}),$$

where $\mathcal{P}$ is a class of endpoints, $\mathcal{R}$ is a class of routes, $s, t : \mathcal{R} \to \mathcal{P}$ assign source and target, $A \subseteq \mathcal{P} \times \mathcal{P}$ is the class of admissible endpoint pairs, D is a class of endpoint-comparison values, and $C : A \to \mathsf{D}$ assigns endpoint data. We assume $(s(\gamma), t(\gamma)) \in A$ for every $\gamma \in \mathcal{R}$. The endpoint map is

$$e : \mathcal{R} \longrightarrow A \times \mathsf{D}, \qquad e(\gamma) = \big((s(\gamma), t(\gamma)), C(s(\gamma), t(\gamma))\big).$$

The endpoint equivalence relation is

$$\gamma \sim_E \eta \quad \Longleftrightarrow \quad e(\gamma) = e(\eta),$$

and $\pi_E : \mathcal{R} \to \mathcal{R}/{\sim_E}$ is the quotient map. We write $[\gamma]_E$ for the $\sim_E$-equivalence class of γ.

Definition 2.3 (Endpoint recovery and constancy on fibers). *Let $I : \mathcal{R} \to S$ be a route invariant. It is determined by endpoint data, or recoverable from endpoint data, if there exists a function $J : e(\mathcal{R}) \to S$ such that $I = J \circ e$. It is constant on endpoint fibers if, whenever $\gamma \sim_E \eta$, one has $I(\gamma) = I(\eta)$. It is route-dependent if it distinguishes at least two routes with the same source, target, and endpoint datum. A framework includes enough information for I when it contains the distinctions needed to evaluate I.*

Definition 2.4 (Endpoint closure for an invariant). *Let $I : \mathcal{R} \to S$ be a route invariant. The endpoint quotient is* closed for I *if I descends to the endpoint quotient; that is, if there exists a necessarily unique function $\bar{I} : \mathcal{R}/{\sim_E} \to S$ such that*

$$I = \bar{I} \circ \pi_E.$$

A failure of this condition is an endpoint-closure obstruction *for I.*

Proposition 2.5 (Factorization criterion for endpoint data). *For a route invariant $I : \mathcal{R} \to S$, the following are equivalent:*

1. *I is determined by endpoint data;*
2. *the endpoint quotient is closed for I;*
3. *I is constant on endpoint fibers;*
4. *there exists a necessarily unique function $\bar{I} : \mathcal{R}/{\sim_E} \to S$ such that*

$$I = \bar{I} \circ \pi_E.$$

Proof. Items (2) and (4) are identical by Definition 2.4. If $I = J \circ e$, then $\gamma \sim_E \eta$ implies $e(\gamma) = e(\eta)$ and hence $I(\gamma) = I(\eta)$, so I is constant on endpoint fibers. If I is constant on endpoint fibers, define $\bar{I}([\gamma]_E) = I(\gamma)$. This is well-defined precisely because all representatives of $[\gamma]_E$ have the same I-value, and then $I = \bar{I} \circ \pi_E$. Finally, if $I = \bar{I} \circ \pi_E$, define $J(e(\gamma)) = \bar{I}([\gamma]_E)$. This is well-defined because equal endpoint data give the same endpoint class, and then $I = J \circ e$. □

Theorem 2.6 (Endpoint-closure criterion). *For a route invariant $I : \mathcal{R} \to S$, the following are equivalent:*

1. *I is recoverable from endpoint data;*
2. *the endpoint quotient is closed for I;*
3. *I factors through the endpoint quotient π_E;*
4. *I is constant on endpoint fibers.*

Consequently, any route invariant that distinguishes routes with the same source, target, and endpoint datum is an endpoint-closure obstruction.

Proof. The equivalences are exactly Proposition 2.5 together with Definition 2.4. If a route invariant distinguishes routes in the same endpoint fiber, then it is not constant on that fiber. It therefore cannot factor through endpoint data, so endpoint closure for that invariant fails. □

Remark 2.7 (Local meaning of endpoint closure)**.** *The term endpoint closure is used here only in the quotient-descent sense. Given a projection $q : X \to Q$ and a quantity $F : X \to S$, say that Q is closed for F when F is recoverable from Q, equivalently when $F = \bar{F} \circ q$ for some $\bar{F} : Q \to S$. If q is surjective, such $\bar{F}$ is unique. Under the specialization $X = \mathcal{R}$, $Q = \mathcal{R}/{\sim_E}$, $q = \pi_E$, and $F = I$, Theorem 2.6 is exactly this quotient-recovery criterion. The map $e : \mathcal{R} \to e(\mathcal{R})$ has the same fibers as π_E, so recovery from endpoint data and descent to the endpoint quotient are equivalent. Thus endpoint closure is a relative property of one map and one invariant. It is not a claim about physically closed systems, dynamics, or completeness of a state space. All claims in this paper follow from quotient descent for the endpoint projection.*

Proposition 2.8 (Endpoint closure as quotient recovery)**.** *Let $q : X \to Q$ be a quotient or projection and let $F : X \to S$ be a quantity. Recovery of F along q is the condition that there exists $\bar{F} : Q \to S$ such that $F = \bar{F} \circ q$. Endpoint closure is exactly the special case*

$$(X, Q, q, F) = (\mathcal{R}, \mathcal{R}/{\sim_E}, \pi_E, I).$$

Thus an endpoint-closure obstruction is exactly failure of the factorization $I = \bar{I} \circ \pi_E$. When the invariant is transport and reversed routes are available, this obstruction is equivalently detected by a loop with nontrivial transport; in smooth transport geometries, the infinitesimal form of local loop transport is curvature. No completeness or additional closure condition is assumed.

Proof. The first statement is the definition of recovery by factorization. Substituting $X = \mathcal{R}$, $Q = \mathcal{R}/{\sim_E}$, $q = \pi_E$, and $F = I$ yields Definition 2.4. Failure of endpoint closure is therefore failure of this factorization. The loop test is Theorem 3.7; the curvature statement is the smooth transport interpretation in Section 5.3. □

Principle 2.9 (Criterion for projection to endpoint data)**.** *Endpoint comparison data determine exactly the invariants constant on endpoint fibers. Equivalently, endpoint projection is closed exactly for those invariants. Every invariant that varies inside an endpoint fiber requires additional route or transport structure.*

The endpoint quotient is the relevant factorization target. Before quotienting there may be paths, histories, morphisms, transports, representatives, or loops. After quotienting there are endpoint classes. Only information constant on endpoint fibers factors through. In this sense, only such information is closed under endpoint projection.

3 Composition, Loops, and Defect

Composition provides a finite test of whether endpoint data suffice. With a direct route $p \to r$ and a composite route $p \to q \to r$, the endpoint pair is the same but the route data may differ. The triangle is a simple compositional case in which endpoint data may have discarded route information.

Definition 3.1 (Transport comparison structure)**.** *A transport comparison structure consists of a class of endpoints $\mathcal{P}$, a class of routes $\mathcal{R}$ with source and target maps $s, t : \mathcal{R} \to \mathcal{P}$, identity routes $1_p : p \to p$, a partially defined associative composition $\eta \circ \gamma : p \to r$ whenever $\gamma : p \to q$ and $\eta : q \to r$, data spaces E_p over endpoints, and maps*

$$\tau_\gamma : E_p \to E_q$$

for every route $\gamma : p \to q$, such that

$$\tau_{1_p} = \mathrm{id}_{E_p}, \qquad \tau_{\eta\circ\gamma} = \tau_\eta \circ \tau_\gamma.$$

When reversed routes are admitted, each route $\gamma : p \to q$ has a route $\gamma^{-1} : q \to p$ with $\tau_{\gamma^{-1}} = \tau_\gamma^{-1}$.

Proposition 3.2 (Endpoint data do not determine route comparison)**.** *Pairwise endpoint data alone do not determine whether direct comparison from p to r agrees with comparison through q.*

Proof. It suffices to exhibit two transport structures with the same endpoint data and different direct-versus-routed values. Let the endpoint data be fixed and constant on all admissible pairs. Take endpoints p, q, r and routes $a : p \to q$, $b : q \to r$, a direct route $c : p \to r$, and the composite route $d = b \circ a : p \to r$. In both structures let every fiber be the same two-element set $E = \{0, 1\}$ and set $\tau_a = \tau_b = \mathrm{id}_E$, hence $\tau_d = \tau_b \circ \tau_a = \mathrm{id}_E$. In the first structure set $\tau_c = \mathrm{id}_E$, so $\tau_c = \tau_d$. In the second let τ_c be the transposition of E, so $\tau_c \neq \tau_d$. The endpoint assignment is the same in both structures, but the direct-versus-composite comparison has a different truth value. Hence that truth value is not determined by endpoint data alone. □

Corollary 3.3 (Triangle criterion)**.** *If a framework distinguishes the direct route $p \to r$ from the composite comparison $p \to q \to r$, then the distinguished quantity is not determined by endpoint data alone.*

Proof. The direct route and the composite route have the same source and target, hence the same endpoint datum $C(p, r)$ under the endpoint map. If an invariant distinguishes them, it is nonconstant on an endpoint fiber. The conclusion follows from Theorem 2.6. □

Principle 3.4 (Direct-versus-composite comparison)**.** *The triangle is a simple compositional test of projection to comparison data: direct route versus composite route. A nontrivial distinction between the two is already a failure of descent to endpoint data.*

Proposition 3.5 (Recovery of transport from endpoint data is path independence)**.** *In a transport comparison structure, view* $\gamma \mapsto \tau_\gamma$ *as a map to the disjoint union of all transport-map sets. It is determined by endpoint data if and only if transport is path independent: for all routes* $\gamma, \eta : p \to q$ *with the same source and target, one has* $\tau_\gamma = \tau_\eta$*.*

Proof. Endpoint data group routes with the same source and target. The invariant $\gamma \mapsto \tau_\gamma$ factors through endpoint data exactly when it is constant on each family of routes with the same source and target. □

When a return route is available, the triangle can be completed to a loop. A loop has degenerate endpoint data: source and target are both p. If transport around the loop is not the identity, the resulting invariant is not determined by the endpoint pair (p, p).

Definition 3.6 (Transport around a loop and holonomy)**.** *For a loop* $\gamma : p \to p$ *in a transport comparison structure, define the transport around the loop by*

$$\delta_p(\gamma) = \tau_\gamma \in \mathrm{Aut}(E_p),$$

when τ_γ *is invertible. The holonomy at* p *is*

$$\mathrm{Hol}_p = \{\tau_\gamma \in \mathrm{Aut}(E_p) : \gamma : p \to p \textit{ is a loop}\}.$$

The transport around the loop is nontrivial if $\tau_\gamma \neq \mathrm{id}_{E_p}$*.*

Theorem 3.7 (Loop test for endpoint-closure obstruction)**.** *Assume transports compose and invert along reversed routes. Let*

$$S_\tau = \bigsqcup_{p,q} \mathrm{Map}(E_p, E_q),$$

and let $I : \mathcal{R} \to S_\tau$ *be given by* $I(\gamma) = \tau_\gamma$*. Then the following are equivalent:*

1. *endpoint closure fails for* I*;*
2. I *is not recoverable from endpoint data;*
3. *transport is route-dependent;*
4. *there exists a loop with nontrivial transport.*

Thus, when reversed routes are available, every endpoint-closure obstruction for transport has a loop test.

Proof. The equivalence of (1), (2), and (3) follows from Theorem 2.6 and Proposition 3.5. Here endpoint data are the source–target data of the transport structure. It remains to compare (3) and (4). If transport is route-dependent, choose routes $\gamma_1, \gamma_2 : p \to q$ with the same source and target and with $\tau_{\gamma_1} \neq \tau_{\gamma_2}$. The loop $\gamma = \gamma_2^{-1} \circ \gamma_1 : p \to p$ has transport

$$\tau_\gamma = \tau_{\gamma_2}^{-1} \circ \tau_{\gamma_1}.$$

If this were id_{E_p}, then $\tau_{\gamma_1} = \tau_{\gamma_2}$, a contradiction. Conversely, a loop $\gamma : p \to p$ with $\tau_\gamma \neq \mathrm{id}_{E_p}$ differs from the constant loop 1_p while sharing the same endpoint data. Thus transport is route-dependent. □

Corollary 3.8 (Holonomy is not determined by endpoint data)**.** *Any framework with nontrivial holonomy contains holonomy information not recoverable from endpoint data alone.*

Proof. A nontrivial loop $\gamma : p \to p$ and the constant loop 1_p have the same endpoint data but different transports. Hence holonomy is nonconstant on an endpoint fiber and is not determined by endpoint data. □

Thus the chain is a sequence of tests. Projection to endpoint data defines the quotient. Composition tests endpoint closure. The triangle is a simple compositional test. A loop is the case with identical source and target. Holonomy is loop information not determined by endpoint data.

4 Standard Structures Used

The factorization result is the quotient factorization statement from Proposition 2.5. The cited references provide standard interpretations of routes, endpoint quotients, and route invariants.

The quotient fact is standard: a function descends through a quotient exactly when it is constant on equivalence classes (Munkres, 2000). In category theory the corresponding data are objects, morphisms, identities, and associative composition (Eilenberg and Mac Lane, 1945; Mac Lane, 1998). In differential geometry the corresponding data are connections, parallel transport, holonomy, and curvature; Wilson loops are closed-loop gauge observables; the Ambrose–Singer theorem relates holonomy to curvature (Ehresmann, 1951; Wilson, 1974; Ambrose and Singer, 1953; Kobayashi and Nomizu, 1963; Nakahara, 2003). In general relativity the corresponding hierarchy is metric–connection– curvature–contraction (Einstein, 1916; Wald, 1984; Malament, 2012; Carroll, 2019).

In each case, the relevant test is to specify the routes, specify the endpoint quotient, and check constancy of the invariant on endpoint fibers.

Proposition 4.1 (External instance schema)**.** *Suppose a cited theory contains a class of routes, an endpoint-data quotient on those routes, and a route invariant I distinguishing some routes with the same endpoint data. Then the failure of I to be determined by endpoint data follows formally from Theorem 2.6.*

Proof. Interpret $\mathcal{R}$, $\sim_E$, and I in the cited theory. If there are routes $\gamma \sim_E \eta$ with $I(\gamma) \neq I(\eta)$, then I is nonconstant on a fiber of π_E. By Proposition 2.5, I cannot factor through endpoint data. □

The cited ingredients used here are quotient descent (Munkres, 2000); category language with distinct parallel arrows (Eilenberg and Mac Lane, 1945; Mac Lane, 1998); connections (Ehresmann, 1951); closed-loop gauge observables (Wilson, 1974); the curvature–holonomy relation and differential-geometric background (Ambrose and Singer, 1953; Kobayashi and Nomizu, 1963); holonomy/path structures in general relativity and Yang–Mills theory (Barrett, 1991); and the relativistic ordering from metric to connection to curvature and the Einstein tensor (Wald, 1984; Malament, 2012; Carroll, 2019).

5 Instances Where Endpoint Data Do Not Determine the Invariant

In each case there is a domain-specific invariant. The criterion above states the condition under which that invariant fails to factor through endpoint data.

5.1 Quotients and Categorical Morphisms

A quotient groups representatives. A quantity descends to the quotient exactly when it is independent of the chosen representative (Munkres, 2000).

Proposition 5.1 (Quotient descent as recovery from endpoint data)**.** *Let $q : X \to X/\!\sim$ be a quotient map. A function $F : X \to S$ is determined by quotient data if and only if there exists $\bar{F} : X/\!\sim \to S$ such that $F = \bar{F} \circ q$. Hence any representative-dependent F is not determined by quotient data.*

Proof. The statement is the quotient descent criterion. If $F = \bar{F} \circ q$, then equivalent representatives have the same value. Conversely, if F is constant on equivalence classes, define $\bar{F}([x]) = F(x)$. This is well-defined exactly because F is constant on each equivalence class. □

The categorical version is compositional. Objects are endpoints; morphisms are routes; parallel morphisms are routes with the same source and target. Object data do not determine morphism-dependent information (Eilenberg and Mac Lane, 1945; Mac Lane, 1998).

Proposition 5.2 (Object data do not determine morphism invariants)**.** *Let $\mathcal{C}$ be a category and let $I : \mathrm{Mor}(\mathcal{C}) \to S$ be a morphism invariant. Let $o : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})^2$ be the map sending a morphism to its ordered source–target pair. Then I is determined by object data alone, meaning $I = J \circ o$ for some J, if and only if I is constant on every hom-class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. In particular, any invariant distinguishing parallel morphisms $f, g : X \to Y$ is not determined by object data.*

Proof. For fixed objects X, Y, the corresponding fiber of o is $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The result is the same fiber-constancy criterion as Proposition 2.5. □

5.2 Gauge Transport and Network Path Products

In gauge theory, two identifications are easily confused. Gauge invariance removes dependence on representative. It does not remove dependence on route. Wilson loops are gauge-invariant loop data, not endpoint data (Wilson, 1974). Connections supply the transport along paths (Ehresmann, 1951; Nakahara, 2003).

Proposition 5.3 (Nontrivial gauge holonomy is not determined by endpoint data)**.** *Let a gauge theory assign parallel transport U_γ to paths $\gamma : p \to q$, and let endpoint data group paths with the same source and target. If a gauge-invariant quantity I built from the transport, for example a Wilson-loop value after completing paths to loops, has different values on two paths with the same source and target, then I is not determined by endpoint data alone.*

Proof. The two paths lie in one endpoint fiber. Since I has different values on them, it is not constant on endpoint fibers. By Proposition 2.5, it cannot factor through endpoint data. □

Networks provide a finite model of the criterion. Vertices are endpoints. Edges carry labels. Paths multiply labels. No independent network theorem is used here; the example is a finite instance of the factorization criterion above.

Proposition 5.4 (Network path products are determined by endpoint data exactly under path independence)**.** *Let a directed network have edge labels in a monoid M, and let $P(\gamma)$ be the ordered product of labels along a finite path γ, with $P(1_p)$ equal to the unit of M. Let endpoint data on paths be their ordered source–target pair. Then P is determined by endpoint data if and only if all paths with the same source and target have the same product. In particular, any nontrivial loop product $P(\gamma) \neq P(1_p)$ is not determined by endpoint data.*

Proof. The endpoint fibers are exactly the families of paths with the same source and target. By Proposition 2.5, the path product factors through endpoints exactly when it is constant on those fibers. □

5.3 Differential Geometry and General Relativity

A connection geometry provides a smooth instance of the same factorization problem. Points are endpoints. Tangent spaces or bundle fibers are data spaces. A connection is the rule for comparing fibers along paths (Ehresmann, 1951; Kobayashi and Nomizu, 1963; Nakahara, 2003). Holonomy is transport around loops. According to the Ambrose–Singer theorem, the infinitesimal generators of holonomy are curvature data transported back to the base point (Ambrose and Singer, 1953; Kobayashi and Nomizu, 1963). Barrett treats holonomy/path structures in general relativity and Yang–Mills theory (Barrett, 1991).

In general relativity the same order appears in Lorentzian form. The metric provides pointwise comparison. Compatible torsion-free transport is Levi-Civita transport. Curvature is formed from that connection. Ricci and scalar contractions enter the Einstein tensor (Einstein, 1916; Wald, 1984; Malament, 2012; Carroll, 2019). The hierarchy is not endpoint–curvature, but endpoint comparison, transport, loop obstruction, curvature, and contraction.

Proposition 5.5 (Nontrivial holonomy is not determined by endpoint data)**.** *In a smooth connection geometry, if parallel transport has nontrivial holonomy, then the corresponding holonomy invariant is not determined by endpoint data. For local or restricted holonomy, the standard Ambrose–Singer curvature–holonomy relation determines the holonomy algebra from curvature data transported back to the base point. Curvature is therefore the smooth infinitesimal form of this failure of descent to endpoint data. This statement does not identify every global holonomy effect with curvature.*

Proof. A nontrivial loop $\gamma : p \to p$ and the constant loop 1_p have the same endpoint data but different transports. Thus holonomy is not determined by endpoint data. For local or restricted holonomy, the Ambrose–Singer theorem determines the holonomy algebra from curvature data transported back to the base point. Thus curvature is the infinitesimal version of this route-dependent level; global flat holonomy is a separate route-dependent obstruction. □

Curvature records the infinitesimal transport obstruction; global flat holonomy records a non-infinitesimal route obstruction.

6 General Endpoint-Closure Obstruction

In each example, the obstruction is nonconstancy on endpoint fibers.

Theorem 6.1 (Endpoint-closure obstruction)**.** *Let a comparison framework contain endpoint data, composable route data, and a route invariant I capable of distinguishing routes with the same endpoint data. Then endpoint closure fails for I. Equivalently, I is not determined by endpoint data, I is not constant on endpoint fibers, and I does not factor through the endpoint quotient π_E. Consequently, endpoint comparison* *data determine only invariants constant on endpoint fibers; representation of I requires additional route or transport structure.*

Proof. If I distinguishes routes γ and η with the same endpoint data, then $\pi_E(\gamma) = \pi_E(\eta)$ but $I(\gamma) \neq I(\eta)$. Thus I is nonconstant on an endpoint fiber. By Theorem 2.6, this is exactly failure of endpoint closure for I, equivalently failure of factorization through π_E. □

Proposition 6.2 (Preservation under fiber-preserving translation)**.** *Let $\mathfrak{C}$ and $\mathfrak{D}$ be comparison frameworks with route classes $\mathcal{R}_{\mathfrak{C}}$ and $\mathcal{R}_{\mathfrak{D}}$ and endpoint equivalence relations $\sim_E^{\mathfrak{C}}$ and $\sim_E^{\mathfrak{D}}$. Let $T : \mathcal{R}_{\mathfrak{C}} \to \mathcal{R}_{\mathfrak{D}}$ be a translation such that*

$$\gamma \sim_E^{\mathfrak{C}} \eta \quad \Longrightarrow \quad T(\gamma) \sim_E^{\mathfrak{D}} T(\eta).$$

If route invariants $I_{\mathfrak{C}}$ and $I_{\mathfrak{D}}$ satisfy $I_{\mathfrak{C}} = I_{\mathfrak{D}} \circ T$, and if endpoint closure fails for $I_{\mathfrak{C}}$, then endpoint closure fails for $I_{\mathfrak{D}}$ on the image of T.

Proof. Since endpoint closure fails for $I_{\mathfrak{C}}$, by Theorem 2.6 there are routes $\gamma \sim_E^{\mathfrak{C}} \eta$ with $I_{\mathfrak{C}}(\gamma) \neq I_{\mathfrak{C}}(\eta)$. The fiber condition on T implies $T(\gamma) \sim_E^{\mathfrak{D}} T(\eta)$, while preservation of invariant values implies $I_{\mathfrak{D}}(T(\gamma)) \neq I_{\mathfrak{D}}(T(\eta))$. Hence $I_{\mathfrak{D}}$ is nonconstant on an endpoint fiber of the translated image. By Theorem 2.6, endpoint closure fails there. □

Gauge holonomy, categorical morphism dependence, network path products, and Riemann curvature are distinct structures. In each case, the same formal endpoint-closure obstruction is present: nonconstancy on endpoint fibers.

7 Scope of the Criterion

The endpoint-closure criterion is not a derivation of connections, gauge fields, metrics, curvature tensors, or equations of motion. It is a limitation result: endpoint data are too coarse whenever the relevant invariant is not constant on endpoint fibers. Representation of that invariant requires additional domain-specific structure.

Nor are dynamics derived. Gauge equations, variational principles, conservation laws, Einstein's field equations, and empirical interpretation require assumptions not present in endpoint data. In general relativity, curvature is downstream of transport. It does not imply the Einstein equations.

The relation among these examples is formal rather than explanatory: each case has an invariant that fails to descend along the relevant endpoint projection. The shared structure is nonconstancy on endpoint fibers, not an additional closure axiom.

Nontrivial holonomy is not a matter of endpoint convention under this criterion. If a smooth geometry that includes the invariant has nontrivial holonomy, endpoint data do not suffice for the relevant invariant. For local or restricted holonomy, curvature is the infinitesimal form of that nonconstancy; global flat holonomy remains a separate route-dependent obstruction.

8 Conclusion

The conclusion is the following:

$$\boxed{\text{Endpoint closure holds exactly for invariants constant on fibers.}}$$

Equivalently, invariants determined by endpoint data are exactly the invariants that factor through the quotient by endpoint-data equivalence. Information separating routes with the same source and target is nonconstant on endpoint fibers. Its representation requires additional route or transport structure.

The sequence is:

$$\boxed{\text{Endpoint} \longrightarrow \text{Triangle} \longrightarrow \text{Loop} \longrightarrow \text{Curvature.}}$$

The endpoint map defines the quotient. A direct route together with a composite route forms the triangle. A return route completes the triangle to a loop. In smooth transport geometry, infinitesimal transport around an infinitesimal loop is represented by curvature.

Failure of quotient descent, categorical morphism dependence, network path dependence, gauge holonomy, Wilson loops, differential-geometric holonomy, curvature, and the curvature hierarchy of general relativity all concern the same formal question: which invariants fail to factor through endpoint data, and hence obstruct endpoint closure?

The obstruction is route-dependent information. Endpoint descriptions are limited by the endpoint map. For any framework using only endpoint data, the condition is constancy on endpoint fibers, equivalently endpoint closure for the invariant under consideration.

Author Biography

Chast K. Wolfe is an independent researcher whose work concerns foundational problems in comparison, quotient descent, and physical theory.

Statements and Declarations

Funding. No funding was received to assist with the preparation of this manuscript.

Competing interests. The author has no relevant financial or non-financial interests to disclose.

Data availability. Data sharing is not applicable because no datasets were generated or analyzed during the preparation of this manuscript.

Use of AI-assisted technologies. Language-editing software was used for copy-editing and consistency checks. The author reviewed and approved the manuscript and takes full responsibility for its content.

Ethics approval. Not applicable.

Consent to participate. Not applicable.

Consent for publication. Not applicable.

Materials and code availability. Not applicable.

Author contributions. Chast K. Wolfe is the sole author of the manuscript and is responsible for the conception, formal development, writing, and final approval of the work.

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