Rectangular Completeness Encompasses Standard Physical Closure

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Abstract

We prove that *closure as rectangular completeness*, developed in the comparison-world framework of [1], stands in a precise relation to standard physical notions of closed two-subsystem systems. For classical Hamiltonian systems, rectangular completeness is exactly equivalent to the full-product component of standard closure. The thermodynamic preparation-space analogue follows as a direct corollary. For quantum systems, rectangular completeness captures exactly the product-state sector, while entangled states already furnish a concrete obstruction on the full density-matrix state space. Because rectangular completeness is formulated purely in terms of binary comparison predicates, with no topology, metric, Hamiltonian, or dynamics presupposed, it strictly extends the standard closure notions on every sector admitting a genuine product-state description. Every classical system satisfying the full-product condition used here, every thermodynamic preparation space of the kind considered here, and every quantum product sector induces a rectangularly complete comparison world, though some rectangularly complete comparison worlds correspond to no physical system with dynamics. Rectangular completeness therefore emerges both as the exact structural characterization of the full-product aspect of classical closure and as the minimal generalization of that notion to abstract relational comparison data.

Keywords Rectangular completeness, comparison worlds, closed systems, thermodynamic closure, product states, entanglement

1 Introduction

The notion of a *closed system* is foundational across all branches of physics. In classical mechanics a system is closed when its state space is the full Cartesian product of its subsystem state spaces and its Hamiltonian involves no coupling to external degrees of freedom. In thermodynamics a closed system exchanges no matter (and an isolated system exchanges neither matter nor energy) with its surroundings. In quantum mechanics a closed system evolves unitarily under a Hamiltonian that couples only internal degrees of freedom. Despite their differences in setting, all three notions share a common structural content: *nothing is missing and nothing enters from outside*.

The paper [1] introduces an alternative definition of closure in the abstract setting of *comparison worlds*. A comparison world (U, C) consists of a set of joint states U and a family of binary predicates C on U; no dynamics, topology, or background geometry is assumed. Closure is identified with *rectangular completeness*: every admissible pairing of a left-profile equivalence class and a right-profile equivalence class is realized by exactly one state. This condition forces a canonical product decomposition $U \cong X_A \times X_B$ and a diagonal symmetry-group action, from which quotient semantics [1, Thm. 3.2] and the failure of subsystem attribution [1, Thm. 7.1] are derived. This paper should therefore be read as a sequel to [1]: we assume the comparison-world machinery developed there and show how standard physical closure fits within that framework.

The present paper answers the natural foundational question:

> *How does rectangular completeness relate to the standard physical definitions of closure? Is it exactly equivalent, or does it strictly generalize them?*

The answer is twofold. For classical Hamiltonian systems equipped with their natural *cross-subsystem* comparison predicates, rectangular completeness is *exactly equivalent* to the full-product component of standard closure (Theorem 5.1). The thermodynamic equivalence follows as a corollary (Corollary 5.4), and the same equivalence holds for the product-state sector of quantum mechanics (Theorem 5.7). At the same time, rectangular completeness is *strictly more general* than the standard physical closure notions considered here: it applies to finite discrete comparison worlds with no dynamics whatsoever (Section 6). These results together show that rectangular completeness provides the canonical structural extension of physical closure to the comparison-world setting.

The equivalence has non-trivial discriminatory content. A system with a kinematic constraint—such as two particles connected by a rigid rod—is typically classified as internally constrained and therefore "closed," yet its joint state space is a proper subset of the full product and the comparison world is not rectangularly complete (Example 5.3). Rectangular completeness correctly identifies such constraints as external couplings on the subsystem-level reading adopted here, precisely as the framework of [1] classifies them: unrealized profile-class pairs are the witnesses of openness. The thermodynamic analogue is equally sharp: a preparation space is rectangularly complete, but the equilibrium manifold imposed by a diathermal wall is not (Example 5.6).

Organization. Section 2 recalls the three standard definitions of closure. Section 3 reviews the comparison-world machinery of [1]. Section 4 introduces cross-subsystem comparison predicates and proves the profile-factorization lemma. Section 5 states and proves the main results. Section 6 establishes strict generality. Section 7 concludes.

2 Standard Notions of Physical Closure

We recall the three standard definitions in enough detail to make the comparison with rectangular completeness precise. Throughout, a *two-subsystem system* consists of two distinguished components A and B.

2.1 Classical Hamiltonian systems

Let X_A and X_B be the state spaces (phase spaces) of subsystems A and B respectively.

Definition 2.1 (Standard classical closure [3, Ch. 8])**.** A two-subsystem classical Hamiltonian system is *standardly closed* if:

(i) the joint state space is the full Cartesian product, $U_{\mathrm{cl}} = X_A \times X_B$; and

(ii) the Hamiltonian $H\colon X_A \times X_B \to \mathbb{R}$ involves no coupling to any external degree of freedom—H is a function of the joint state alone, with no additional environment variable.

Condition (i) is the *completeness* condition: every combination of an A-state and a B-state is physically accessible. Condition (ii) is the *non-coupling* condition: evolution is internally determined. Condition (i) is the operative one for the structural theorems of this paper; condition (ii) concerns dynamics, which the comparison-world framework does not assume.

Remark 2.2 (Open systems and missing states). When a system is open—coupled to an external environment—the effective joint state space is generically a proper subset $U_{\mathrm{eff}} \subsetneq X_A \times X_B$. External coupling imposes constraints that forbid certain (x_A, x_B) combinations; the comparison-world counterpart is a failure of rectangular completeness in which some (α-class, β-class) pairs are unrealized.

2.2 Thermodynamic systems

Let X_A^{th} and X_B^{th} denote the spaces of thermodynamic states of components A and B respectively. A thermodynamic state records macrostate variables (internal energy, volume, particle numbers, etc.) sufficient to determine the component's thermodynamic properties.

Definition 2.3 (Standard thermodynamic closure [4, Ch. 1–2])**.** A two-component thermodynamic system is:

- *closed* if no matter is exchanged with the external environment;
- *isolated* if neither matter nor energy is exchanged with the external environment.

In the subsystem-level preparation setting considered here, the *preparation state space*—the set of all independently preparable joint macrostates, prior to any constrained equilibrium—is the full product $U_{\mathrm{th}} = X_A^{\mathrm{th}} \times X_B^{\mathrm{th}}$.

Remark 2.4 (Preparation versus equilibrium). The *preparation state space* (all independently attainable initial macrostates) must be distinguished from the *equilibrium manifold* (states satisfying equilibrium conditions, e.g., equal temperatures or pressures). The equilibrium manifold is a proper subset of the preparation state space. The comparison-world construction is applied here to the preparation state space, which is the relevant full-product state space for the subsystem-level comparison developed in this paper, because the comparison predicates of Section 4 compare arbitrary joint states, not merely equilibrium states.

2.3 Quantum systems

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces for subsystems A and B, with joint Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Definition 2.5 (Standard quantum closure [5, § 2.2])**.** A bipartite quantum system is *standardly closed* if it evolves unitarily under a Hamiltonian H on $\mathcal{H}_A \otimes \mathcal{H}_B$ with no coupling to an external system.

The full state space $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ (density matrices) is not a Cartesian product of the individual-subsystem state spaces when entanglement is present. Section 5.3 shows that rectangular completeness captures the product-state sector and that marginal-based comparison data do not distinguish all states on the full state space.

3 Comparison Worlds and Rectangular Completeness

We recall the essential machinery from [1], where all proofs appear.

Definition 3.1 (Comparison world [1, Def. 2.1])**.** A *comparison world* is a pair (U, C) where U is a nonempty set of states and C is a set of binary predicates $c\colon U \times U \to \{0, 1\}$, in the sense of relational structures [6]. No topology, metric, manifold structure, or background geometry is assumed.

For each $u \in U$ define the *left profile* $\mathsf{L}(u)\colon C \times U \to \{0, 1\}$ by $\mathsf{L}(u)(c, w) = c(u, w)$, and the *right profile* $\mathsf{R}(u)\colon C \times U \to \{0, 1\}$ by $\mathsf{R}(u)(c, w) = c(w, u)$. Define equivalence relations α and β on U by

$$u\,\alpha\,v \iff \mathsf{L}(u) = \mathsf{L}(v), \qquad u\,\beta\,v \iff \mathsf{R}(u) = \mathsf{R}(v). \tag{1}$$

Set $X_A := U/\alpha$, $X_B := U/\beta$, and define the *canonical factor map*

$$\Theta\colon U \to X_A \times X_B, \qquad \Theta(u) := \big([u]_\alpha,\ [u]_\beta\big). \tag{2}$$

Definition 3.2 (Rectangular completeness [1, Def. 2.6])**.** The comparison world (U, C) is *rectangularly complete* if for every $A \in X_A$ and every $B \in X_B$ there exists a *unique* $u \in U$ with $[u]_\alpha = A$ and $[u]_\beta = B$. Equivalently, the canonical factor map Θ is a bijection $U \xrightarrow{\sim} X_A \times X_B$.

Remark 3.3 (Internal reading). Rectangular completeness says there are no *missing* states and no *redundant* states relative to the product $X_A \times X_B$ determined by the comparison data alone. Each pair of profile classes is realized by exactly one state. This is the purely internal expression of "nothing is missing and nothing enters from outside."

The following results are proved in [1].

Theorem 3.4 (Diagonal action theorem [1, Thm. 2.8])**.** *If (U, C) is rectangularly complete, then Θ is a bijection $U \cong X_A \times X_B$, and $G = \mathrm{Aut}(U, C)$ acts diagonally on $X_A \times X_B$. The orbit quotient $\pi\colon X_A \times X_B \to \mathsf{Phys} := (X_A \times X_B)/G$ is canonically determined.*

Theorem 3.5 (Minimality [1, Thm. 2.9])**.** *Rectangular completeness is the minimal internal condition that forces Θ to be bijective across all comparison worlds.*

Theorem 3.6 (Asymmetry [1, Cor. 2.11])**.** *If (U, C) is rectangularly complete and $|U| > 1$, then C contains at least one asymmetric predicate: there exist $c \in C$ and $u, w \in U$ with $c(u, w) \neq c(w, u)$.*

Theorem 3.6 shows that any genuinely two-subsystem comparison world must contain at least one asymmetric predicate. This motivates the class of predicates introduced in the next section.

4 Cross-Subsystem Comparison Predicates

The key to relating rectangular completeness to standard physical closure is the choice of comparison predicates. We introduce a canonical class that is both physically natural and structurally necessary.

Definition 4.1 (Observable separation)**.** Let Z be a set and $\mathcal{F}$ a family of functions $f\colon Z \to \mathbb{R}$. We say $\mathcal{F}$ *separates* Z if for any $z \neq z'$ in Z there exists $f \in \mathcal{F}$ with $f(z) \neq f(z')$.

Remark 4.2 (Constant observables). Throughout we assume the observable families contain all constant functions $z \mapsto t$, $t \in \mathbb{R}$. This allows comparison against arbitrary scalar thresholds without affecting which subsystem states are separated or identified.

Definition 4.3 (Cross-subsystem predicate)**.** Let $U = X_A \times X_B$, and let $\mathcal{F}_A$ (resp. $\mathcal{F}_B$) be a family of functions $X_A \to \mathbb{R}$ (resp. $X_B \to \mathbb{R}$). For each pair $(f, g) \in \mathcal{F}_A \times \mathcal{F}_B$, the *cross-subsystem comparison predicate* $c_{f,g}$ is

$$c_{f,g}\big((u_A, u_B), (v_A, v_B)\big) \ :=\ \mathbf{1}\big[f(u_A)\ \geq\ g(v_B)\big]. \tag{3}$$

The family of all such predicates, $C_{\mathrm{cs}} := \{c_{f,g} : f \in \mathcal{F}_A,\ g \in \mathcal{F}_B\}$, is the *cross-subsystem comparison family* generated by $(\mathcal{F}_A, \mathcal{F}_B)$.

Remark 4.4 (Physical naturality). A cross-subsystem predicate $c_{f,g}$ compares an A-observable of the left state with a B-observable of the right state. Representative examples: the kinetic energy of A in state u versus the potential energy of B in state v; the position of particle A along axis i in u versus the momentum of particle B along axis j in v; the internal energy of thermodynamic component A in u versus the entropy of component B in v. These are the natural *relational comparisons between the two subsystems*: they record how the A-content of one state relates to the B-content of another, and are generically asymmetric—reversing the arguments exchanges the role of A and B observables with no algebraic reason for the values to agree.

Remark 4.5 (Why this family is canonical). The cross-subsystem family C_{cs} is canonical for three reasons: it is built directly from subsystem observables with no joint-system input; it compares the A-content of one state with the B-content of another, thereby resolving the two subsystem coordinates independently; and it supplies the asymmetry required by Theorem 3.6. It is the leanest observable-generated comparison language that accomplishes all three.

The next lemma is the structural key to all main results.

Lemma 4.6 (Profile factorization)**.** *Let $U = Y_A \times Y_B$ and let $C = C_{\mathrm{cs}}$ be the cross-subsystem family generated by separating families $\mathcal{F}_A$ on Y_A and $\mathcal{F}_B$ on Y_B, each containing all constant functions. Then the following hold.*

(i) The left profile $\mathsf{L}(u)$ depends only on the A-component: if $u_A = v_A$, then $\mathsf{L}(u) = \mathsf{L}(v)$.

(ii) The right profile $\mathsf{R}(u)$ depends only on the B-component: if $u_B = v_B$, then $\mathsf{R}(u) = \mathsf{R}(v)$.

(iii) The congruence α is determined by the A-component:

$$(u_A, u_B)\, \alpha\, (v_A, v_B) \iff u_A \sim_{\mathcal{F}_A} v_A.$$

(iv) The congruence β is determined by the B-component:

$$(u_A, u_B)\, \beta\, (v_A, v_B) \iff u_B \sim_{\mathcal{F}_B} v_B.$$

(v) Consequently, $U/\alpha \cong Y_A/{\sim_{\mathcal{F}_A}}$ and $U/\beta \cong Y_B/{\sim_{\mathcal{F}_B}}$. When $\mathcal{F}_A$ separates Y_A (and $\mathcal{F}_B$ separates Y_B), both quotients are trivial: $U/\alpha \cong Y_A$ and $U/\beta \cong Y_B$.

Proof. (i). For any $c_{f,g} \in C_{\mathrm{cs}}$ and $w \in U$, $c_{f,g}(u, w) = \mathbf{1}[f(u_A) \geq g(w_B)] = \mathbf{1}[f(v_A) \geq g(w_B)] = c_{f,g}(v, w)$, using $u_A = v_A$. Since $c_{f,g}$ was arbitrary, $\mathsf{L}(u) = \mathsf{L}(v)$.

(ii). Similarly, $c_{f,g}(w, u) = \mathbf{1}[f(w_A) \geq g(u_B)] = \mathbf{1}[f(w_A) \geq g(v_B)] = c_{f,g}(w, v)$ using $u_B = v_B$, so $\mathsf{R}(u) = \mathsf{R}(v)$.

(iii). $u\, \alpha\, v$ means $c_{f,g}(u, w) = c_{f,g}(v, w)$ for all f, g, w. Fix $f \in \mathcal{F}_A$; for each $t \in \mathbb{R}$ let g_t be the constant function with value t. Then $\mathbf{1}[f(u_A) \geq t] = c_{f,g_t}(u, w) = c_{f,g_t}(v, w) = \mathbf{1}[f(v_A) \geq t]$ for all t. If $f(u_A) \neq f(v_A)$, choose t strictly between the two values; the two indicator functions then differ at t, a contradiction. Hence $f(u_A) = f(v_A)$. Since f was arbitrary, $u_A \sim_{\mathcal{F}_A} v_A$. Conversely, $u_A \sim_{\mathcal{F}_A} v_A$ gives $c_{f,g}(u, w) = c_{f,g}(v, w)$ for all g, w, so $\mathsf{L}(u) = \mathsf{L}(v)$.

(iv). Fix $g \in \mathcal{F}_B$; use constant functions $f_t \in \mathcal{F}_A$ and the identity $c_{f_t,g}(w, u) = \mathbf{1}[t \geq g(u_B)]$. The same threshold argument as in (iii) shows that $u\, \beta\, v$ implies $u_B \sim_{\mathcal{F}_B} v_B$. Conversely, if $u_B \sim_{\mathcal{F}_B} v_B$, then $c_{f,g}(w, u) = c_{f,g}(w, v)$ for all f, w, so $\mathsf{R}(u) = \mathsf{R}(v)$.

(v). Immediate from (iii)–(iv). When $\mathcal{F}_A$ separates Y_A, $\sim_{\mathcal{F}_A}$ is equality, so $U/\alpha \cong Y_A$; similarly for U/β. □

5 Main Results

5.1 Classical Hamiltonian systems

Theorem 5.1 (Classical equivalence)**.** *Let X_A and X_B be the phase spaces of subsystems A and B. Let $\mathcal{F}_A$ and $\mathcal{F}_B$ be separating families of observables on X_A and X_B respectively, each containing all constant observables, and let $C = C_{\mathrm{cs}}$. Assume every subsystem state is realized by some joint state: $\mathrm{pr}_A(U) = X_A$ and $\mathrm{pr}_B(U) = X_B$.*

A two-subsystem classical system with joint state space $U \subseteq X_A \times X_B$ and comparison predicates $C \restriction U$ is rectangularly complete if and only if $U = X_A \times X_B$, i.e., the system satisfies the full-product condition in Definition 2.1(i).

Proof. ($\Leftarrow$) Full product implies rectangularly complete. Suppose $U = X_A \times X_B$. By Lemma 4.6(v), $U/\alpha \cong X_A$ and $U/\beta \cong X_B$. Under these identifications $\Theta\colon U \to X_A \times X_B$ sends $(u_A, u_B) \mapsto (u_A, u_B)$, which is the identity map and hence a bijection. Rectangular completeness holds.

($\Rightarrow$) Rectangularly complete implies full product. Suppose $(U, C \restriction U)$ is rectangularly complete; suppose for contradiction that $U \subsetneq X_A \times X_B$. Pick $(a, b) \in (X_A \times X_B) \setminus U$. By the surjectivity assumptions choose $(a, b') \in U$ and $(a', b) \in U$. Set

$$A_a := [(a, b')]_\alpha \in U/\alpha, \qquad B_b := [(a', b)]_\beta \in U/\beta.$$

By Lemma 4.6(iii)–(iv) these classes depend only on a and b, not on a' or b'. If some $u = (u_A, u_B) \in U$ realized the pair (A_a, B_b), then the same lemma would give $u_A = a$ and $u_B = b$, hence $u = (a, b) \notin U$, a contradiction. Thus (A_a, B_b) is an unrealized pair in $U/\alpha \times U/\beta$, contradicting rectangular completeness. Hence $U = X_A \times X_B$. □

Remark 5.2 (Exact equivalence, not mere implication). The theorem establishes a biconditional between rectangular completeness and the full-product condition in the standard classical definition. The surjectivity assumptions are normalization conditions, not extra closure hypotheses: they identify X_A and X_B as the subsystem state spaces actually present in the system under study. With that understood, the two definitions express the same structural content in different languages.

Example 5.3 (Constrained system: the biconditional is non-trivial)**.** Consider two classical particles A and B moving on the real line and joined by a rigid rod of fixed length $L > 0$. The rod is typically classified as an internal constraint, and one might therefore call the two-particle system "closed." However, the rod forces $q_B - q_A = L$, so the effective joint state space is

$$U_{\text{rod}} = \{(q,\, q + L) : q \in \mathbb{R}\} \subsetneq \mathbb{R} \times \mathbb{R} = X_A \times X_B.$$

With cross-subsystem observables $f = g = \text{id}$, the α-congruence on U_{rod} identifies states by their A-position and β by their B-position, giving $U_{\text{rod}}/\alpha \cong \mathbb{R}$ and $U_{\text{rod}}/\beta \cong \mathbb{R}$. The canonical factor map is $\Theta(q, q + L) = (q, q + L) \in \mathbb{R} \times \mathbb{R}$, whose image is the affine diagonal $\{(q, q + L) : q \in \mathbb{R}\}$. Any pair (q_A, q_B) with $q_B - q_A \neq L$ is unrealized; for instance, the pair $(0, 0)$ has no representative in U_{rod}. Rectangular completeness fails, and Theorem 5.1 establishes $U_{\text{rod}} \subsetneq X_A \times X_B$: on the subsystem-level reading used throughout this paper, the rod is a mechanical linkage external to A and external to B, and its constraint on their pairing is the signature of an open system at the subsystem level. In the language of [1], the unrealized (α, β) pairs are *witnesses of openness* (Definition 10.1 of [1]): the profile-class evidence that the system reaches outside itself to determine which states of A are compatible with which states of B.

5.2 Thermodynamic systems

Corollary 5.4 (Thermodynamic equivalence)**.** *Let X_A^{th} and X_B^{th} be the thermodynamic state spaces of components A and B, let $\mathcal{F}_A^{\text{th}}$ and $\mathcal{F}_B^{\text{th}}$ be separating families of thermodynamic observables (internal energy, entropy, volume, particle numbers, etc.) each containing all constant observables, and let $C_{\text{cs}}^{\text{th}}$ be the induced cross-subsystem family.*

A two-component thermodynamic system with preparation state space $U^{\text{th}} \subseteq X_A^{\text{th}} \times X_B^{\text{th}}$, satisfying $\text{pr}_A(U^{\text{th}}) = X_A^{\text{th}}$ and $\text{pr}_B(U^{\text{th}}) = X_B^{\text{th}}$, is rectangularly complete (with respect to $C_{\text{cs}}^{\text{th}}$) if and only if $U^{\text{th}} = X_A^{\text{th}} \times X_B^{\text{th}}$, i.e., its preparation state space has the full-product form associated here with a closed or isolated two-component system. In particular, the equilibrium manifold of any two-component system placed in thermal or mechanical contact is never rectangularly complete.

Proof. Theorem 5.1 applies with $X_A, X_B, \mathcal{F}_A, \mathcal{F}_B$ replaced by $X_A^{\text{th}}, X_B^{\text{th}}, \mathcal{F}_A^{\text{th}}, \mathcal{F}_B^{\text{th}}$. The surjectivity hypotheses play the same normalizing role as in the classical theorem. The "in particular" clause holds because any equilibrium constraint (equal temperatures, pressures, or chemical potentials) restricts the preparation state space to a proper subset of the product, which by the necessity direction leaves profile-class pairs unrealized. □

Remark 5.5 (Preparation versus equilibrium). The corollary applies to the preparation state space, not the equilibrium manifold. The preparation state space of a closed or isolated system is the full product: any macrostate of A and any macrostate of B can be independently prepared before being brought together. Once an external element—a diathermal wall, a moveable partition, a semipermeable membrane—mediates contact, it restricts which macrostate pairings are accessible and RC fails, as the following example makes precise.

Example 5.6 (Oxygen and nitrogen: preparation versus contact)**.** Let A be oxygen and B nitrogen, each in its own insulated container with macrostate (T, V, N). Before any contact, any combination of macrostates is independently preparable: $U^{\text{th}} = X_A^{\text{th}} \times X_B^{\text{th}}$ is the full product, and Corollary 5.4 confirms rectangular completeness.

Insert a rigid diathermal wall (thermal contact, no matter or volume exchange). The equilibrium manifold is

$$U_{\text{eq}} = \{(T, V_A, N_A,\, T, V_B, N_B) : T, V_i > 0,\ N_i \in \mathbb{N}\} \subsetneq X_A^{\text{th}} \times X_B^{\text{th}}.$$

With the cross-subsystem predicate comparing T_A to T_B, the α-classes are indexed by temperature of A and the β-classes by temperature of B. Any profile-class pair (T_A, T_B) with $T_A \neq T_B$ has no representative in U_{eq}; rectangular completeness fails. The wall is the external coupling: before its insertion the system is genuinely closed at the subsystem level; the wall's introduction is recorded in the comparison world exactly as the appearance of unrealized profile-class pairs.

5.3 Quantum systems

The Hilbert-space tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ contains entangled states not of the product form $\psi_A \otimes \psi_B$. We treat the product-state sector and the full state space separately.

5.3.1 Product-state sector

Let $S_A := \{\psi_A : \|\psi_A\| = 1\}/U(1)$ and $S_B := \{\psi_B : \|\psi_B\| = 1\}/U(1)$ be the projective state spaces of $\mathcal{H}_A$ and $\mathcal{H}_B$, and let

$$U_{\mathrm{prod}} := \{[\psi_A \otimes \psi_B] : [\psi_A] \in S_A,\ [\psi_B] \in S_B\} \ \cong\ S_A \times S_B.$$

Let $\mathcal{F}_A^Q$ be the family of pure-state expectation-value functionals $[\psi_A] \mapsto \langle \psi_A, O_A \psi_A \rangle$ as O_A ranges over bounded self-adjoint operators on $\mathcal{H}_A$, and $\mathcal{F}_B^Q$ the analogous family for $\mathcal{H}_B$. These families separate their respective projective state spaces by the completeness of the observable algebra, and contain constant functionals via scalar operators λI.

Theorem 5.7 (Quantum product-sector equivalence). *Let C_{cs}^Q be the cross-subsystem family on $U_{\mathrm{prod}} \cong S_A \times S_B$ generated by $(\mathcal{F}_A^Q, \mathcal{F}_B^Q)$. Then $(U_{\mathrm{prod}}, C_{\mathrm{cs}}^Q)$ is rectangularly complete, and the canonical decomposition recovers $X_A \cong S_A$, $X_B \cong S_B$.*

Proof. $U_{\mathrm{prod}} = S_A \times S_B$ is a full product with separating observable families, so Theorem 5.1 applies directly with S_A and S_B in place of X_A and X_B. □

5.3.2 Entangled states and the boundary of the framework

Proposition 5.8 (Full quantum state space is not rectangularly complete). *Let $U_Q = \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. For self-adjoint O_A on $\mathcal{H}_A$ and O_B on $\mathcal{H}_B$, define*

$$c_{O_A,O_B}(\rho, \sigma) := \mathbf{1}\big[\mathrm{Tr}(O_A \rho_A) \geq \mathrm{Tr}(O_B \sigma_B)\big],$$

where $\rho_A = \mathrm{Tr}_B \rho$ and $\sigma_B = \mathrm{Tr}_A \sigma$. Let C be the family of all such predicates. Then (U_Q, C) is not rectangularly complete whenever $\dim \mathcal{H}_A \geq 2$ and $\dim \mathcal{H}_B \geq 2$.

Proof. If $\rho_A = \sigma_A$ then $c_{O_A,O_B}(\rho, \tau) = c_{O_A,O_B}(\sigma, \tau)$ for all O_A, O_B, τ, so $\rho\, \alpha\, \sigma$. Conversely, if $\rho\, \alpha\, \sigma$, choose $O_B = tI$ to obtain $\mathbf{1}[\mathrm{Tr}(O_A \rho_A) \geq t] = \mathbf{1}[\mathrm{Tr}(O_A \sigma_A) \geq t]$ for all t. If $\mathrm{Tr}(O_A \rho_A) \neq \mathrm{Tr}(O_A \sigma_A)$, choosing t strictly between the two values gives different indicators, a contradiction. Hence $\mathrm{Tr}(O_A \rho_A) = \mathrm{Tr}(O_A \sigma_A)$ for every self-adjoint O_A, and therefore $\rho_A = \sigma_A$. So $\rho\, \alpha\, \sigma \iff \rho_A = \sigma_A$; by symmetry, $\rho\, \beta\, \sigma \iff \rho_B = \sigma_B$. The canonical factor map $\Theta \colon \rho \mapsto (\rho_A, \rho_B)$ is not injective: the Bell state

$$\rho := |\Phi^+\rangle\langle\Phi^+|, \quad |\Phi^+\rangle := \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}},$$

and the product state $\sigma := \frac{1}{2}(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|) \otimes \frac{1}{2}(|0\rangle_B\langle 0| + |1\rangle_B\langle 1|)$ satisfy $\rho_A = \sigma_A = \frac{1}{2} I_A$ and $\rho_B = \sigma_B = \frac{1}{2} I_B$, yet $\rho \neq \sigma$. Hence Θ is not injective and rectangular completeness fails. □

Remark 5.9 (Entanglement as additional structure). Proposition 5.8 delineates the boundary of what the comparison-world framework captures: the framework describes the product structure of a two-subsystem world, while global correlation structure, including entanglement, lies beyond what marginal profiles encode. The broader theoretical development in [2] treats entanglement-type enrichment as part of the morphism-level transport structure identified by the two-locus exhaustion theorem (Theorem 9.10 of [1]).

6 Strict Generality of Rectangular Completeness

The results of Section 5 show that rectangular completeness contains the standard classical and thermodynamic closure conditions considered here, together with the quantum product-sector case. We now show that it is *strictly* more general.

Example 6.1 (Finite discrete closed world)**.** Let $X_A = \{a_1, a_2\}$, $X_B = \{b_1, b_2\}$, $U = X_A \times X_B$. Define

$$\begin{aligned} c_1\big((u_A, u_B), (v_A, v_B)\big) &= \mathbf{1}[u_A = a_1] \cdot \mathbf{1}[v_B = b_2], \\ c_2\big((u_A, u_B), (v_A, v_B)\big) &= \mathbf{1}[u_A = a_2] \cdot \mathbf{1}[v_B = b_1]. \end{aligned}$$

These are cross-subsystem predicates with $f_1 = \mathbf{1}_{a_1}$, $g_1 = \mathbf{1}_{b_2}$, $f_2 = \mathbf{1}_{a_2}$, $g_2 = \mathbf{1}_{b_1}$. The left profiles distinguish a_1 from a_2 and the right profiles distinguish b_1 from b_2; hence α identifies states by their A-component and β by their B-component. All four (α, β) pairs are realized exactly once. The comparison world $(U, \{c_1, c_2\})$ is rectangularly complete; it has no topology, no metric, no Hamiltonian, and no dynamics, and corresponds to no classical, thermodynamic, or quantum system.

Proposition 6.2 (Strict inclusion)**.** *Let $\mathcal{RC}$ denote the class of all rectangularly complete comparison worlds, and let $\mathcal{PC}$ denote the class of two-subsystem systems that are closed in the standard classical or thermodynamic senses, together with the product-state sectors of standardly closed bipartite quantum systems. When $\mathcal{PC}$ is embedded into $\mathcal{RC}$ via its natural comparison world (Section 4):*

(i) $\mathcal{PC} \subseteq \mathcal{RC}$*;*

(ii) $\mathcal{PC} \subsetneq \mathcal{RC}$*.*

Proof. (i) is the content of Theorem 5.1, Corollary 5.4, and Theorem 5.7. (ii) follows from Example 6.1, which is in $\mathcal{RC}$ but has no physical dynamics and therefore is not in $\mathcal{PC}$. □

Remark 6.3 (What strict generality means). From the perspective of physics, rectangular completeness applies where standard physical notions are not defined: finite relational systems, combinatorial state spaces, logical models. It isolates the structure of closure from the physical machinery surrounding it. From the perspective of mathematics, rectangular completeness is the unique minimal condition (Theorem 3.5) forcing the canonical product decomposition $U \cong X_A \times X_B$ from pure comparison data. The physical definitions of closure are *realizations* of that structural condition in specific contexts.

7 Conclusion

We have proved that the comparison-world definition of closure (rectangular completeness) stands in the following relation to the standard physical definitions:

(a) *Classical Hamiltonian systems.* Rectangular completeness is exactly equivalent to the full-product component of standard closure. The two definitions express the same structural content; neither implies the other for free. (Theorem 5.1, Example 5.3.)

(b) *Thermodynamic systems.* The same equivalence holds on preparation state spaces. The equilibrium manifold of a system in thermal or mechanical contact is never rectangularly complete. (Corollary 5.4, Example 5.6.)

(c) *Quantum product sector.* Rectangular completeness captures the product-state sector of a closed bipartite quantum system. Marginal-based comparison data do not separate the full quantum state space; entangled states already furnish a concrete obstruction to rectangular completeness there. (Theorem 5.7, Proposition 5.8.)

(d) *Strict generality.* Rectangular completeness applies to comparison worlds with no dynamics, metric, or topology, forming a class strictly larger than the standard closed systems considered here. (Proposition 6.2.)

The composite result: *rectangular completeness is exactly equivalent to the structural full-product condition in the classical and thermodynamic cases, and the minimal structural extension of that notion across the full class of comparison worlds.*

These results justify the use of rectangular completeness as the closure criterion in [1]: every two-subsystem classical or thermodynamic system satisfying the full-product condition considered here, and every quantum product sector of a standardly closed bipartite system, induces a rectangularly complete comparison world, and rectangular completeness is the minimal condition with that property.

Broader implications. The exact structural equivalence in the classical case means that the results of [1]—quotient semantics (Thm. 3.2), the failure of subsystem attribution (Thm. 7.1), and the two-locus exhaustion theorem (Thm. 9.10)—apply to every classical two-subsystem system whose joint state space is the full product, without additional assumptions. The thermodynamic case shows the same structural picture governs preparation spaces before equilibrium constraints are imposed. The quantum case shows these results apply on the product-state sector of closed bipartite quantum systems, with global correlation structure, including entanglement, representing the residual enrichment identified by the two-locus theorem.

Statements and Declarations

Competing Interests The author declares that there are no competing interests.

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