Closed Systems from Comparison Completeness

A Theory of Closed Relational Systems

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Preface

This book is a theorem-first monograph on closed relational systems. Its objective is precise: start from intrinsic comparison data alone, assume no background geometry or external frame, and classify exactly which global structures are determined.

The work is not a reinterpretation of standard geometric physics in new language. It is a closure test. At each stage, a structure is admitted only if it is internally recoverable from the relational system itself. What survives that test forms the admissible core of the theory.

The resulting architecture is rigid. Closure determines quotient semantics, extension freedom collapses to two loci, transport obstruction first stabilizes in a quadratic layer, and smooth realization turns that same layer into curvature. The later geometric and field-theoretic chapters therefore appear as downstream consequences of the same intrinsic closure mechanism.

The manuscript is organized as a dependency chain. Early chapters prove compactness and semantic closure; middle chapters classify enrichment and obstruction; later chapters derive dynamics, curvature, causal scaling, Hilbert structure, Einstein compatibility, connection-first reconstruction, and electromagnetic quantization. The appendix then shows the same determined scalar channel in a concrete confinement setting.

For readers who want the exact dependency map before technical details, Chapter 0 provides the complete theorem-level roadmap.

Contents

Introduction

A mathematically closed physical theory must be written without external scaffolding. If observers, clocks, laboratories, and comparison protocols are all internal to the same world being modeled, then no background manifold, state space linearity, causal order, or field content can be accepted as primitive by convention alone. Such structure is admissible only when it is recoverable from intrinsic relational data.

This manuscript develops that principle as a classification program. Starting from binary comparison structure alone, it asks which global constructions are determined, which are excluded, and where any residual freedom can still occur. The central claim is that the admissible architecture is highly rigid: once closed-world admissibility is imposed, quotient semantics, transport structure, the first stable obstruction layer, and the repaired intrinsic–smooth interface are mathematically constrained, and geometric, field-theoretic, scalar, and realization-level sectors emerge downstream from those constraints.

Guiding question. Given only intrinsic comparison data on a closed world, which mathematical structures are necessary consequences rather than auxiliary choices?

0.1 Primitive object and admissibility stance

Definition 0.1.1 (Comparison world)**.** A *comparison world* is a pair

$$(U, \mathcal{C}),$$

where $U \neq \varnothing$ is a set of states and $\mathcal{C}$ is a family of binary comparison predicates

$$c : U \times U \to \{0, 1\}.$$

We require $\mathcal{C}$ to be *intrinsic*: writing $G = \mathrm{Aut}(U, \mathcal{C})$ for the automorphism group of Theorem 2.2.3, every predicate is G-invariant,

$$c(gu, gv) = c(u, v) \qquad \text{for all } c \in \mathcal{C},\ g \in G,\ u, v \in U.$$

This is a fixed-point condition: G is determined by $\mathcal{C}$, and $\mathcal{C}$ is admissible exactly when its predicates are stable under the symmetries they generate. A predicate violating it assigns comparison values tracking a labeling of states that no symmetry of the world preserves; such a predicate is an external evaluator in the sense of (SP1), not an intrinsic comparison. Intrinsicality is therefore the primitive-level expression of the intrinsicity clause (SP1): a comparison is a relation among states, not a stipulated labeling of them.

No topology, differentiable structure, metric, linear structure, probability measure, or background arena is assumed at this level. The admissibility stance of the book is strict: all higher structure must be justified as internally determined by the comparison system and its closed-world coherence conditions.

0.2 Standing principle

Standing Principle 1 (Closed-World Admissibility)**.** Throughout the manuscript we work under one standing principle, unpacked into the following clauses.

(SP1) **Intrinsicity clause.** No background manifold, external observer frame, or auxiliary state-space primitive is admissible unless reconstructed internally from comparison data.

(SP2) **Compactness clause.** Finite-stage and global admissibility are linked by the exact finite-to-global boundary of the refinement tower. Under intrinsicality this boundary is finitary: by Theorem 2.5.12 every comparison separation is witnessed at a single coordinate, so the refinement-tower limit requires no completion and the finite-to-global boundary of Theorems 1.6.4 and 1.8.1 holds without separate hypothesis. See Theorems 2.5.12 and 2.5.13.

(SP3) **Comparison-completeness clause.** Closedness is profile-pair completeness (rectangular completeness). Under intrinsicality (Theorem 0.1.1) and local distinguishability, this is not an independent hypothesis but a consequence; see Theorem 2.6.4.

(SP4) **Quotient-descent clause.** Admissible semantic content is exactly the content descending through intrinsic quotient maps.

(SP5) **Transport-visibility clause.** Any admissible residue beyond quotient semantics is carried by the transport-obstruction hierarchy and its first visible layers.

Remark 0.2.1 (How hypotheses are used). In subsequent chapters, theorem headers use either *local setup hypotheses* (e.g., choosing objects, indices, or connected components) or *realization-level technical hypotheses* (e.g., smooth realizability, jet injectivity). These are not independent foundational axioms; they are scoped instantiations of items (SP1) to (SP5). Any non-local quantitative boundary condition left after theorem-level discharge is recorded explicitly in chapter A.

0.3 Methodological strategy

The derivation is organized as a finite-to-global and algebra-to-geometry program. At finite stage, one controls compatibility through Boolean admissibility; at global stage, one passes through exact compactness boundaries. On the semantic side, comparison completeness determines canonical quotient structure. On the enrichment side, one proves that non-quotient content can occur through at least one of two loci, possibly both, and no third. On the transport side, one isolates a canonical filtration and proves that its first stable visible obstruction is quadratic. Under compatible smooth realization, the stabilized quadratic carrier enters the repaired intrinsic–smooth interface: the degree–2 placement is forced by triangle minimality, commutator filtration, and Stone-limit stabilization, while faithful realization on that forced carrier makes nonzero stabilized square classes equivalent to nonzero realized curvature. The subsequent macroscopic, scalar, and realization-level consequences proceed through that realized-channel package.

At a conceptual level, the logical progression is

$$\begin{aligned}(U,\mathcal{C}) &\Longrightarrow \text{admissibility boundary and canonical quotient semantics}\\ &\Longrightarrow \text{two-locus enrichment and transport filtration}\\ &\Longrightarrow \text{first stable quadratic obstruction}\\ &\Longrightarrow \text{repaired intrinsic–smooth interface and macroscopic compatibility laws}\\ &\Longrightarrow \text{scalar-channel classification, numerical closure, and canonical realization.}\end{aligned}$$

0.4 Main structural claim

Theorem 0.4.1 records the structural content proved across chapters 1 to 22.

Theorem 0.4.1 (Structural rigidity of closed relational systems)**.** *Let $(U,\mathcal{C})$ be a comparison world satisfying the closed-system admissibility and comparison-completeness conditions developed in this manuscript. Then the following structure is determined.*

(1) *Admissible state content factors through canonical quotient semantics, with intrinsic product decomposition $U \cong X_A \times X_B$ and physical quotient $\mathsf{Phys} = (X_A \times X_B)/G$.*

(2) *Beyond quotient semantics, admissible enrichment occurs through at least one of two loci, possibly both—object-level representative choice and morphism-level transport—with no third locus.*

(3) *Transport admits a canonical descending filtration $F^1 \supset F^2 \supset F^3 \supset \cdots$, whose first stable visible obstruction is the quadratic layer F^2/F^3.*

(4) *The same obstruction appears as finite triangle failure, global loop-descent obstruction, and their bridge identification on minimal subsystems.*

(5) *The obstruction governs the internal observer-reduction framework and the conditional regime of compatible irreversible descent, and admits a canonical spacetime interleaving realization via a two-parameter inverse-limit object.*

(6) *Under smooth realization, the stabilized quadratic carrier admits an intrinsic–smooth interface: the refinement tower is proved to stabilize its first nontrivial transport obstruction at degree* 2*, and faithful smooth realization on that forced carrier makes a nonzero stabilized square class equivalent to nonzero realized curvature on the realized degree–*2 *channel. This degree–*2 *realized-channel package feeds causal, Hilbert, Einstein-boundary, connection, electromagnetic, matter-sector, and numerical-closure consequences.*

(7) *The scalar channel determined by the quadratic carrier is then classified intrinsically, assigned its canonical matter-sector realization role, and finally yields a concrete nonperturbative invariant in the magnetic-confinement appendix.*

Proof. Each item is proved in the indicated chapter block of the manuscript. Item (1) follows from the compactness-to-quotient chain in chapters 1 to 4. Item (2) is the two-locus classification proved in chapters 5 and 6. Items (3) and (4) are established by the transport obstruction chain in chapters 7 to 9. Item (5) is derived in chapters 10 and 11. Item (6) is proved through the smooth-realization chapter, the repaired intrinsic-to-smooth interface, and their macroscopic continuations in chapters 12 to 21. Item (7) is completed by the scalar-classification and realization chain in chapters B and 19 to 22. Thus the theorem is exactly the consolidated statement of the chapterwise results listed above. □

The point of the theorem is methodological as much as structural: it does not append geometry, field theory, or matter-sector realization to an independently chosen background. It shows how these sectors arise as closure-compatible realizations of structure already determined at the relational level, with scalar and realization-level consequences forced downstream of the same obstruction chain.

0.5 What is proved, and what is not assumed

The manuscript proves theorem-level necessity results under the explicit admissibility hypotheses developed in the text. It does not assume the standard ordering "geometry first, fields second." Instead, it proves a reverse ordering: semantic closure and transport obstruction first, then geometric, field-theoretic, scalar, and realization-level structure, and only afterward any appendix-level application.

The admissibility hypotheses of this manuscript reduce to a single primitive condition. By Theorems 2.5.11 and 2.5.12 and Theorem 2.6.4, the compactness clause (SP2) and the comparison-completeness clause (SP3) are consequences of intrinsicality (Theorem 0.1.1) together with local distinguishability. Intrinsicality is not a hypothesis about the universe but a condition on what counts as a comparison: that a predicate be invariant under the symmetries it helps constitute. The program therefore rests on one identity—to be a distinction is to be a symmetry-invariant of comparison—which is the intrinsicity clause (SP1) stated as a definition of the primitive rather than a prohibition on later constructions. The manuscript is not axiom-free; it has exactly one axiom, and that axiom is the meaning of comparison.

In this sense, the contribution is not only a new derivation of familiar objects. It is a sharpened criterion for admissibility in closed-system modeling: which constructions remain valid when all external reference structures are removed.

0.6 Organization of the manuscript

The manuscript is organized as a dependency-driven argument rather than a collection of independent chapters.

Foundational phase. Chapters 1 to 4 establish the finite-to-global admissibility boundary, derive comparison completeness, force quotient semantics, and isolate the subsystem-attribution boundary. This phase fixes the semantic backbone of the closed system.

Classification and obstruction phase. Chapters 5 to 9 prove the universal two-locus enrichment classification and identify the first nontrivial transport obstruction through finite and global representatives of the same defect.

Dynamical and geometric phase. Chapters 10 to 14 translate the obstruction framework into the internal observer-reduction framework, the conditional regime of compatible irreversible descent, spacetime interleaving, curvature realization, and macroscopic causal structure.

Macroscopic compatibility and field-theoretic phase. Chapters 15 to 18 derive determined Hilbert structure, the Einstein compatibility boundary, connection-first reconstruction, and electromagnetic phase/charge structure from the same intrinsic chain.

Scalar-sector and numerical-closure phase. Chapters 19 to 22 classify primitive matter sectors, derive the quartic depth law and intrinsic scalar closure, and determine the canonical realization role of the scalar invariant before any appendix-level application.

Concrete downstream realization. Chapter B then shows that the scalar channel already fixed by the main stack yields a concrete nonperturbative adiabatic invariant in magnetic confinement.

0.7 How to read this book

A natural first reading is theorem-driven: chapters 1, 4, 5, 9, 13, 16, 18, 19 and 21. This path isolates the main boundary and rigidity statements with minimal technical detours.

A second reading is construction-driven: chapters 2, B, 6, 8, 10 to 12, 17, 20 and 22. This path foregrounds explicit constructions and realization mechanisms.

Both readings are equivalent in conclusion: for closed relational systems, the admissible global architecture is determined far more rigidly than conventional background-first formulations suggest.

0.8 Conclusion

This introduction states the manuscript's governing theorem program in exact form: within a closed world, admissible structure must be derived from intrinsic comparison data and not posited as external background. The chapter architecture is therefore a proof architecture, ordered from compactness and quotient semantics through transport obstruction to geometric, field-theoretic, scalar, and realization-level closure.

Accordingly, chapter 1 is logically first: it determines the finite-to-global admissibility boundary that is reused by every later construction.

Part I

Foundations

Chapter 1

Refinement Towers and the Quantifier-Reversal Boundary

1.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the compactness clause item (SP2) in explicit Boolean form. It isolates the mechanism governing the passage from finite-stage admissibility to global admissibility. Its role in the sequel is foundational: it provides the exact obstruction principle used later to pass from finite comparison data to global comparison structure, and in particular underlies the Boolean rectangle-splitting arguments of chapter 2. Consider an increasing tower of finite-stage Boolean descriptions of a system together with compatible exclusion ideals recording forbidden local patterns. Because each stage ideal is proper, every finite stage admits admissible ultrafilters, that is, ultrafilters avoiding the excluded sets. The basic question is whether these finite-stage admissible objects assemble coherently into a genuine admissible object at the limit. The obstruction is exact and is characterized by Theorem 1.6.4. For an exclusion tower, each stage ideal is proper, so Theorem 1.4.3 guarantees admissible ultrafilters at every finite stage. The genuine question is whether these stagewise admissible ultrafilters assemble into a global admissible ultrafilter on B_∞. This happens if and only if no finite family of excluded sets from the tower already covers the whole universe.

Equivalently, under the standing exclusion-tower hypotheses,

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) \neq \varnothing \quad \Longleftrightarrow \quad \text{there do not exist } E_1, \dots, E_n \in \bigcup_{B \in \mathbb{N}} \mathsf{J}_B \text{ with } U = \bigcup_{i=1}^{n} E_i.$$

Everything in this chapter is purely Boolean-algebraic and realizes only the compactness layer of the standing principle. No product structure, symmetry group, quotient semantics, transport data, or smooth realization is invoked here; those belong to later clauses items (SP3) to (SP5). Throughout this chapter,

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

1.2 Refinement towers

Definition 1.2.1 (Refinement tower)**.** A *refinement tower* consists of

(1) a nonempty set U;

(2) for each $B \in \mathbb{N}$, a Boolean algebra

$$\mathsf{B}_B \subseteq \mathcal{P}(U),$$

viewed as a Boolean subalgebra of $\mathcal{P}(U)$ with unit U;

(3) inclusions

$$\mathsf{B}_B \subseteq \mathsf{B}_{B+1} \qquad \text{for all } B \in \mathbb{N}.$$

We write

$$\mathsf{B}_\infty := \bigcup_{B \in \mathbb{N}} \mathsf{B}_B.$$

Lemma 1.2.2 (The direct union is already Boolean)**.** *For a refinement tower, the union*

$$\mathsf{B}_\infty = \bigcup_{B \in \mathbb{N}} \mathsf{B}_B$$

is a Boolean subalgebra of $\mathcal{P}(U)$*.*

Proof. Let $A, C \in \mathsf{B}_\infty$. Choose $B_A, B_C \in \mathbb{N}$ such that

$$A \in \mathsf{B}_{B_A}, \qquad C \in \mathsf{B}_{B_C}.$$

Set

$$B := \max\{B_A, B_C\}.$$

Since the tower is increasing,

$$A, C \in \mathsf{B}_B.$$

Because B_B is a Boolean algebra, one has

$$A \cup C \in \mathsf{B}_B, \qquad A \cap C \in \mathsf{B}_B, \qquad U \setminus A \in \mathsf{B}_B.$$

Hence all three sets belong to B_∞. Therefore B_∞ is closed under finite unions, finite intersections, and complements, and is thus a Boolean subalgebra of $\mathcal{P}(U)$. ☐

Remark 1.2.3 (Direct-limit viewpoint). Because the tower is increasing, the direct limit of the Boolean algebras $(\mathsf{B}_B)_B$ is literally the union B_∞. No additional Boolean completion is required.

1.3 Exclusion towers

Definition 1.3.1 (Exclusion tower)**.** Let

$$\big(U, (\mathsf{B}_B)_{B\in\mathbb{N}}\big)$$

be a refinement tower. An *exclusion tower* over it is a family of ideals

$$\mathsf{J}_B \subseteq \mathsf{B}_B \qquad (B \in \mathbb{N})$$

such that

(1) each J_B is proper;

(2) the family is compatible under restriction:

$$\mathsf{J}_B = \mathsf{J}_{B+1} \cap \mathsf{B}_B \qquad \text{for all } B \in \mathbb{N}.$$

Lemma 1.3.2 (Iterated compatibility)**.** *If $B \le C$, then*

$$\mathsf{J}_B = \mathsf{J}_C \cap \mathsf{B}_B.$$

Proof. The case $C = B + 1$ is part of Theorem 1.3.1. The general case follows by induction on $C - B$. ☐

Definition 1.3.3 (Limit ideal)**.** The associated limit ideal is

$$\mathsf{J}_\infty := \left\langle \bigcup_{B\in\mathbb{N}} \mathsf{J}_B \right\rangle_{\mathsf{B}_\infty},$$

that is, the ideal of B_∞ generated by the union of the stage ideals.

Lemma 1.3.4 (Concrete description of the limit ideal)**.** *Let*

$$\mathcal{S} := \bigcup_{B\in\mathbb{N}} \mathsf{J}_B.$$

Then

$$\mathsf{J}_\infty = \left\{ A \in \mathsf{B}_\infty : \exists n \ge 1,\ \exists E_1, \dots, E_n \in \mathcal{S} \text{ such that } A \subseteq \bigcup_{i=1}^{n} E_i \right\}.$$

In particular,

$$U \in \mathsf{J}_\infty \iff \exists n \ge 1,\ \exists E_1, \dots, E_n \in \mathcal{S} \text{ such that } U = \bigcup_{i=1}^{n} E_i.$$

Proof. Define

$$\mathcal{I} := \left\{ A \in \mathsf{B}_\infty : \exists n \geq 1,\ \exists E_1, \dots, E_n \in \mathcal{S} \text{ such that } A \subseteq \bigcup_{i=1}^{n} E_i \right\}.$$

We show that $\mathcal{I}$ is exactly the ideal generated by $\mathcal{S}$. First, $\mathcal{I}$ is an ideal of B_∞. Certainly

$$\varnothing \in \mathcal{I},$$

since $\varnothing \subseteq E$ for every $E \in \mathcal{S}$. If $A \in \mathcal{I}$ and $A' \in \mathsf{B}_\infty$ satisfies $A' \subseteq A$, then the same witnessing family shows $A' \in \mathcal{I}$. Thus $\mathcal{I}$ is downward closed. If $A, C \in \mathcal{I}$, choose witnesses

$$A \subseteq \bigcup_{i=1}^{m} E_i, \qquad C \subseteq \bigcup_{j=1}^{n} F_j,$$

with $E_i, F_j \in \mathcal{S}$. Then

$$A \cup C \subseteq \left(\bigcup_{i=1}^{m} E_i \right) \cup \left(\bigcup_{j=1}^{n} F_j \right),$$

so $A \cup C \in \mathcal{I}$. Hence $\mathcal{I}$ is an ideal. Next, $\mathcal{S} \subseteq \mathcal{I}$, since every $S \in \mathcal{S}$ satisfies $S \subseteq S$. Now let $\mathcal{K}$ be any ideal of B_∞ containing $\mathcal{S}$. If $A \in \mathcal{I}$, choose $E_1, \dots, E_n \in \mathcal{S}$ such that

$$A \subseteq \bigcup_{i=1}^{n} E_i.$$

Since $\mathcal{K}$ is an ideal containing each E_i, it contains their finite union and hence, by downward closure, also A. Therefore $\mathcal{I} \subseteq \mathcal{K}$. Thus $\mathcal{I}$ is the smallest ideal of B_∞ containing $\mathcal{S}$, so $\mathcal{I} = \mathsf{J}_\infty$. The final equivalence follows by taking $A = U$. Since each $E_i \subseteq U$, the condition

$$U \subseteq \bigcup_{i=1}^{n} E_i$$

is equivalent to

$$U = \bigcup_{i=1}^{n} E_i.$$

□

1.4 Ultrafilters and admissibility

Definition 1.4.1 (Ultrafilters and admissibility)**.** For a Boolean algebra B, let $\mathrm{UF}(\mathsf{B})$ denote the set of ultrafilters on B. If $\mathsf{J} \subseteq \mathsf{B}$ is an ideal, define

$$\mathrm{UF}(\mathsf{B}; \mathsf{J}) := \{ \mathfrak{u} \in \mathrm{UF}(\mathsf{B}) : \mathfrak{u} \cap \mathsf{J} = \varnothing \}.$$

Elements of $\mathrm{UF}(\mathsf{B}; \mathsf{J})$ are called *admissible ultrafilters* relative to J.

Definition 1.4.2 (Finite-stage and global admissibility)**.** Finite-stage admissibility at level B means

$$\mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B) \neq \varnothing.$$

Global admissibility means

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) \neq \varnothing.$$

Lemma 1.4.3 (Ideal separation)**.** *Let* B *be a Boolean algebra and* $\mathsf{J} \subseteq \mathsf{B}$ *an ideal. Then*

$$\mathrm{UF}(\mathsf{B}; \mathsf{J}) \neq \varnothing \quad \Longleftrightarrow \quad \mathsf{J} \textit{ is proper.}$$

Equivalently,

$$\mathrm{UF}(\mathsf{B}; \mathsf{J}) = \varnothing \quad \Longleftrightarrow \quad 1_{\mathsf{B}} \in \mathsf{J}.$$

Proof. Assume first that $\mathfrak{u} \in \mathrm{UF}(\mathsf{B}; \mathsf{J})$. Since $\mathfrak{u}$ is an ultrafilter,

$$1_{\mathsf{B}} \in \mathfrak{u}.$$

If $1_{\mathsf{B}} \in \mathsf{J}$, then

$$1_{\mathsf{B}} \in \mathfrak{u} \cap \mathsf{J},$$

contradicting admissibility. Hence J is proper. Conversely, assume J is proper, so that

$$1_{\mathsf{B}} \notin \mathsf{J}.$$

Define

$$\mathcal{F} := \{\neg E : E \in \mathsf{J}\}.$$

We claim that $\mathcal{F}$ has the finite intersection property. Let $E_1, \dots, E_n \in \mathsf{J}$. Since J is an ideal,

$$E := \bigvee_{i=1}^{n} E_i \in \mathsf{J}.$$

Because J is proper,

$$E \neq 1_{\mathsf{B}},$$

and hence

$$\neg E \neq 0_{\mathsf{B}}.$$

By De Morgan's law,

$$\bigwedge_{i=1}^{n} \neg E_i = \neg \left(\bigvee_{i=1}^{n} E_i \right) = \neg E \neq 0_{\mathsf{B}}.$$

So $\mathcal{F}$ has the finite intersection property. Let $\mathfrak{f}$ be the filter generated by $\mathcal{F}$. It is proper. By the ultrafilter extension theorem, there exists an ultrafilter

$$\mathfrak{u} \supseteq \mathfrak{f}.$$

If $E \in \mathsf{J}$, then $\neg E \in \mathcal{F} \subseteq \mathfrak{u}$. Thus $E \notin \mathfrak{u}$, since otherwise

$$E \wedge \neg E = 0_{\mathsf{B}} \in \mathfrak{u},$$

which is impossible for a proper filter. Therefore

$$\mathfrak{u} \cap \mathsf{J} = \varnothing,$$

so $\mathfrak{u} \in \mathrm{UF}(\mathsf{B}; \mathsf{J})$. ∎

1.5 Finite witnesses, first failure, and persistence

Lemma 1.5.1 (Finite witness). *Global admissibility fails if and only if there exist*

$$E_1, \dots, E_n \in \bigcup_{B \in \mathbb{N}} \mathsf{J}_B$$

such that

$$U = \bigcup_{i=1}^{n} E_i.$$

Equivalently,

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) = \varnothing \quad \Longleftrightarrow \quad \exists n \geq 1,\ \exists E_1, \dots, E_n \in \bigcup_{B} \mathsf{J}_B \text{ with } U = \bigcup_{i=1}^{n} E_i.$$

Proof. By Theorem 1.4.3,

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) = \varnothing \quad \Longleftrightarrow \quad \mathsf{J}_\infty \text{ is improper.}$$

Since J_∞ is an ideal in B_∞, this is equivalent to

$$U = 1_{\mathsf{B}_\infty} \in \mathsf{J}_\infty.$$

By Theorem 1.3.4, this holds if and only if there exist $E_1, \dots, E_n \in \bigcup_B \mathsf{J}_B$ such that

$$U \subseteq \bigcup_{i=1}^{n} E_i.$$

Because each $E_i \subseteq U$, this is equivalent to

$$U = \bigcup_{i=1}^{n} E_i.$$

□

Remark 1.5.2. Theorem 1.5.1 shows that global admissibility can fail even when each stage ideal J_B is proper. The obstruction is not local but combinatorial: it arises exactly when finitely many excluded patterns, possibly drawn from different stages, already cover the entire universe.

Lemma 1.5.3 (First failure scale). *If global admissibility fails, then there exists $B_0 \in \mathbb{N}$ and sets*

$$E_1, \dots, E_n \in \mathsf{J}_{B_0}$$

such that

$$U = \bigcup_{i=1}^{n} E_i.$$

Proof. By Theorem 1.5.1, choose

$$E_1, \ldots, E_n \in \bigcup_B \mathsf{J}_B$$

such that

$$U = \bigcup_{i=1}^n E_i.$$

For each i, choose $B_i \in \mathbb{N}$ with $E_i \in \mathsf{J}_{B_i}$, and set

$$B_0 := \max_i B_i.$$

Since the tower is increasing,

$$E_i \in \mathsf{B}_{B_0} \qquad \text{for all } i.$$

By Theorem 1.3.2,

$$\mathsf{J}_{B_i} = \mathsf{J}_{B_0} \cap \mathsf{B}_{B_i} \qquad \text{for all } i,$$

hence

$$E_i \in \mathsf{J}_{B_0} \qquad \text{for all } i.$$

Thus the same finite covering witness already occurs at stage B_0. ☐

Theorem 1.5.4 (Existence of a minimal failure depth)**.** *If global admissibility fails, then the set*

$$\mathcal{B}_{\text{fail}} := \left\{ B \in \mathbb{N} : \exists E_1, \ldots, E_n \in \mathsf{J}_B \textit{ such that } U = \bigcup_{i=1}^n E_i \right\}$$

is nonempty and therefore has a least element.

Proof. By Theorem 1.5.3, the set $\mathcal{B}_{\text{fail}}$ is nonempty. Since it is a nonempty subset of $\mathbb{N}$, it has a least element. ☐

Theorem 1.5.5 (Persistence of failure)**.** *If admissibility fails at stage B_0, then*

$$\mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B) = \varnothing \qquad \textit{for all } B \geq B_0.$$

Proof. Assume there exist

$$E_1, \ldots, E_n \in \mathsf{J}_{B_0}$$

such that

$$U = \bigcup_{i=1}^n E_i.$$

Fix $B \geq B_0$. Because the tower is increasing,

$$E_i \in \mathsf{B}_B \qquad \text{for all } i.$$

By Theorem 1.3.2,

$$\mathsf{J}_{B_0} = \mathsf{J}_B \cap \mathsf{B}_{B_0},$$

hence

$$E_i \in \mathsf{J}_B \qquad \text{for all } i.$$

Therefore

$$U = \bigcup_{i=1}^{n} E_i \in \mathsf{J}_B,$$

so J_B is improper. By Theorem 1.4.3,

$$\mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B) = \varnothing.$$

□

1.6 The quantifier-reversal boundary

Definition 1.6.1 (Coherent ultrafilter tower)**.** A *coherent ultrafilter tower* is a sequence

$$(\mathfrak{u}_B)_{B \in \mathbb{N}}$$

such that

$$\mathfrak{u}_B \in \mathrm{UF}(\mathsf{B}_B) \qquad \text{for all } B,$$

and

$$\mathfrak{u}_{B+1} \cap \mathsf{B}_B = \mathfrak{u}_B \qquad \text{for all } B.$$

It is *admissible* if

$$\mathfrak{u}_B \cap \mathsf{J}_B = \varnothing \qquad \text{for all } B.$$

Lemma 1.6.2 (Union of a coherent ultrafilter tower)**.** *Let*

$$(\mathfrak{u}_B)_{B \in \mathbb{N}}$$

be a coherent ultrafilter tower, and define

$$\mathfrak{u} := \bigcup_{B \in \mathbb{N}} \mathfrak{u}_B \subseteq \mathsf{B}_\infty.$$

Then $\mathfrak{u}$ *is an ultrafilter on* B_∞*, and for every* $B \in \mathbb{N}$*,*

$$\mathfrak{u} \cap \mathsf{B}_B = \mathfrak{u}_B.$$

Proof. We first show that $\mathfrak{u}$ is a proper filter on B_∞. If $A, C \in \mathfrak{u}$, choose B large enough that $A, C \in \mathsf{B}_B$. Then $A, C \in \mathfrak{u}_B$, hence

$$A \cap C \in \mathfrak{u}_B \subseteq \mathfrak{u}.$$

If $A \in \mathfrak{u}$ and $A \subseteq D \in \mathsf{B}_\infty$, choose B large enough that $A, D \in \mathsf{B}_B$. Then $A \in \mathfrak{u}_B$, so by upward closure of the filter $\mathfrak{u}_B$,

$$D \in \mathfrak{u}_B \subseteq \mathfrak{u}.$$

Also $\varnothing \notin \mathfrak{u}$, since $\varnothing \notin \mathfrak{u}_B$ for every B. Thus $\mathfrak{u}$ is a proper filter on B_∞. Next we show that $\mathfrak{u}$ decides every element of B_∞. Let $A \in \mathsf{B}_\infty$. Choose B such that $A \in \mathsf{B}_B$. Since $\mathfrak{u}_B$ is an ultrafilter on B_B, exactly one of A and $U \setminus A$ belongs to $\mathfrak{u}_B$, hence exactly one belongs to $\mathfrak{u}$. Therefore $\mathfrak{u}$ is an ultrafilter on B_∞. Finally, if $A \in \mathfrak{u} \cap \mathsf{B}_B$, then $A \in \mathfrak{u}_C$ for some $C \geq B$. Coherence gives

$$\mathfrak{u}_C \cap \mathsf{B}_B = \mathfrak{u}_B,$$

so $A \in \mathfrak{u}_B$. The reverse inclusion

$$\mathfrak{u}_B \subseteq \mathfrak{u} \cap \mathsf{B}_B$$

is immediate. Hence

$$\mathfrak{u} \cap \mathsf{B}_B = \mathfrak{u}_B \qquad \text{for all } B.$$

□

Lemma 1.6.3 (Admissibility passes to the limit)**.** *Let*

$$(\mathfrak{u}_B)_{B \in \mathbb{N}}$$

be an admissible coherent ultrafilter tower, and let

$$\mathfrak{u} := \bigcup_{B \in \mathbb{N}} \mathfrak{u}_B.$$

Then

$$\mathfrak{u} \in \mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty).$$

Proof. By Theorem 1.6.2, $\mathfrak{u}$ is an ultrafilter on B_∞. It remains to prove admissibility. Let $A \in \mathsf{J}_\infty$. By Theorem 1.3.4, there exist

$$E_1, \ldots, E_n \in \bigcup_B \mathsf{J}_B$$

such that

$$A \subseteq \bigcup_{i=1}^n E_i.$$

Choose B with all $E_i \in \mathsf{J}_B$. Since

$$\mathfrak{u} \cap \mathsf{B}_B = \mathfrak{u}_B$$

by Theorem 1.6.2, and

$$\mathfrak{u}_B \cap \mathsf{J}_B = \varnothing,$$

each $E_i \notin \mathfrak{u}$, hence

$$U \setminus E_i \in \mathfrak{u} \qquad \text{for all } i.$$

Therefore

$$\bigcap_{i=1}^{n} (U \setminus E_i) \in \mathfrak{u}.$$

But

$$\bigcap_{i=1}^{n} (U \setminus E_i) \subseteq U \setminus A,$$

so

$$U \setminus A \in \mathfrak{u},$$

and hence $A \notin \mathfrak{u}$. Thus

$$\mathfrak{u} \cap \mathsf{J}_\infty = \varnothing,$$

so

$$\mathfrak{u} \in \mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty).$$

□

Theorem 1.6.4 (Quantifier-reversal boundary)**.** *The following are equivalent.*

(1)

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) \neq \varnothing.$$

(2) *For every finite family*

$$E_1, \ldots, E_n \in \bigcup_{B \in \mathbb{N}} \mathsf{J}_B,$$

one has

$$\bigcup_{i=1}^{n} E_i \neq U.$$

(3) *There exists an admissible coherent ultrafilter tower.*

Proof. (1)⇒(2). This is the contrapositive of Theorem 1.5.1. *(2)⇒(3).* We construct an admissible coherent tower inductively. For the base step, condition (2) implies that J_0 is proper; otherwise

$$U \in \mathsf{J}_0 \subseteq \bigcup_{B} \mathsf{J}_B,$$

contradicting (2). By Theorem 1.4.3, choose

$$\mathfrak{u}_0 \in \mathrm{UF}(\mathsf{B}_0; \mathsf{J}_0).$$

Assume $\mathfrak{u}_B \in \mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B)$ has been constructed. Define

$$\mathcal{G}_{B+1} := \mathfrak{u}_B \cup \{U \setminus E : E \in \mathsf{J}_{B+1}\} \subseteq \mathsf{B}_{B+1}.$$

We claim that $\mathcal{G}_{B+1}$ has the finite intersection property. Take finitely many

$$A_1, \dots, A_m \in \mathfrak{u}_B, \qquad E_1, \dots, E_n \in \mathsf{J}_{B+1},$$

and set

$$A := \bigcap_{j=1}^{m} A_j \in \mathfrak{u}_B, \qquad E := \bigcup_{i=1}^{n} E_i \in \mathsf{J}_{B+1}.$$

If

$$A \cap (U \setminus E) = \varnothing,$$

then $A \subseteq E$. Since $A \in \mathsf{B}_B \subseteq \mathsf{B}_{B+1}$ and J_{B+1} is downward closed, this implies

$$A \in \mathsf{J}_{B+1} \cap \mathsf{B}_B = \mathsf{J}_B,$$

contradicting $\mathfrak{u}_B \cap \mathsf{J}_B = \varnothing$. Therefore

$$A \cap (U \setminus E) \neq \varnothing.$$

So $\mathcal{G}_{B+1}$ has the finite intersection property. Let $\mathfrak{f}_{B+1}$ be the filter generated by $\mathcal{G}_{B+1}$. It is proper, hence extends to an ultrafilter

$$\mathfrak{u}_{B+1} \in \mathrm{UF}(\mathsf{B}_{B+1}).$$

Since $\mathfrak{u}_B \subseteq \mathfrak{u}_{B+1}$ and $\mathfrak{u}_B$ is already an ultrafilter on B_B, one has

$$\mathfrak{u}_{B+1} \cap \mathsf{B}_B = \mathfrak{u}_B.$$

Also, for every $E \in \mathsf{J}_{B+1}$,

$$U \setminus E \in \mathfrak{u}_{B+1},$$

so $E \notin \mathfrak{u}_{B+1}$. Hence

$$\mathfrak{u}_{B+1} \cap \mathsf{J}_{B+1} = \varnothing.$$

Induction yields an admissible coherent ultrafilter tower. *(3)⇒(1).* This is Theorem 1.6.3. □

1.7 Inverse-limit representation

Theorem 1.7.1 (Inverse-limit representation)**.** *Restriction along the inclusions $\mathsf{B}_B \subseteq \mathsf{B}_{B+1}$ induces a canonical bijection*

$$\mathrm{UF}(\mathsf{B}_\infty) \cong \varprojlim_{B} \mathrm{UF}(\mathsf{B}_B),$$

where the inverse limit is taken with respect to the restriction maps

$$\mathrm{UF}(\mathsf{B}_{B+1}) \to \mathrm{UF}(\mathsf{B}_B).$$

Proof. Define

$$\Phi : \mathrm{UF}(\mathsf{B}_\infty) \to \varprojlim_B \mathrm{UF}(\mathsf{B}_B), \qquad \Phi(\mathfrak{u}) := (\mathfrak{u} \cap \mathsf{B}_B)_B.$$

This is well defined because the restriction of an ultrafilter to a Boolean subalgebra is an ultrafilter, and the restrictions are coherent. Conversely, let

$$(\mathfrak{u}_B)_B \in \varprojlim_B \mathrm{UF}(\mathsf{B}_B).$$

Define

$$\Psi((\mathfrak{u}_B)_B) := \bigcup_B \mathfrak{u}_B \subseteq \mathsf{B}_\infty.$$

By Theorem 1.6.2, this union is an ultrafilter on B_∞, and for every B,

$$\Psi((\mathfrak{u}_B)_B) \cap \mathsf{B}_B = \mathfrak{u}_B.$$

Thus Ψ is a well-defined map

$$\Psi : \varprojlim_B \mathrm{UF}(\mathsf{B}_B) \to \mathrm{UF}(\mathsf{B}_\infty).$$

It is immediate that

$$\Phi(\Psi((\mathfrak{u}_B)_B)) = (\mathfrak{u}_B)_B$$

and

$$\Psi(\Phi(\mathfrak{u})) = \mathfrak{u}.$$

Therefore Φ is a bijection with inverse Ψ. □

Corollary 1.7.2 (Admissible inverse-limit representation)**.** *Restriction induces a canonical bijection*

$$\mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty) \cong \varprojlim_B \mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B).$$

Proof. By Theorem 1.7.1, ultrafilters on B_∞ are in bijection with coherent ultrafilter towers on the stages. If

$$\mathfrak{u} \in \mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty),$$

then for each B,

$$\mathfrak{u} \cap \mathsf{J}_B = \varnothing,$$

because $\mathsf{J}_B \subseteq \mathsf{J}_\infty$. Hence each stage restriction is admissible. Conversely, let $(\mathfrak{u}_B)_B$ be a coherent tower with

$$\mathfrak{u}_B \in \mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B) \qquad \text{for all } B.$$

Then by Theorem 1.6.3,

$$\bigcup_B \mathfrak{u}_B \in \mathrm{UF}(\mathsf{B}_\infty; \mathsf{J}_\infty).$$

Thus the bijection of Theorem 1.7.1 restricts to the claimed bijection on admissible ultrafilters. □

1.8 Stability criterion

Theorem 1.8.1 (Stability of admissibility)**.** *For an exclusion tower satisfying the compactness clause item* (SP2)*, finite-stage admissibility holds at every level, and global admissibility holds if and only if finite non-coverage holds, namely condition* (2) *of Theorem 1.6.4.*

Proof. Because each stage ideal J_B is proper by Theorem 1.3.1, Theorem 1.4.3 gives $\mathrm{UF}(\mathsf{B}_B; \mathsf{J}_B) \neq \varnothing$ for every B. The remaining claim is exactly Theorem 1.6.4, which identifies global admissibility with finite non-coverage without any further hypothesis. □

1.9 Conclusion

The compactness boundary established in Theorems 1.5.1, 1.5.3, 1.6.4 and 1.7.2 is exact: for an exclusion tower, each proper stage ideal already admits admissible ultrafilters, and global admissibility fails if and only if finitely many exclusions from the tower already cover the universe. Equivalently, admissible ultrafilters on the limit Boolean algebra are precisely admissible coherent towers of stage ultrafilters, so once finite non-coverage holds, no further global obstruction remains.

Chapter 2 applies this compactness boundary to comparison worlds and derives the canonical factorization that initiates quotient semantics.

Chapter 2

Comparison Completeness and the Emergence of Diagonal Redundancy

2.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the comparison-completeness clause item (SP3). Building on the compactness boundary of chapter 1, it derives the canonical product decomposition and diagonal-redundancy data used in chapters 3 and 4. Chapter 1 isolated the Boolean compactness principle governing the passage from finite-stage admissibility to global admissibility. In particular, Theorems 1.6.4, 1.7.2 and 1.8.1 show that admissible global objects exist precisely when no finite family of excluded patterns already covers the universe. The present chapter applies this compactness mechanism to the primitive comparison data introduced in the Introduction. Starting from a comparison world

$$(U, \mathcal{C}),$$

we analyze the intrinsic indistinguishability relations determined by the comparison predicates and determine the exact condition under which the comparison system admits a canonical global structure. From the comparison predicates one obtains two intrinsic equivalence relations recording indistinguishability of outgoing and incoming comparison profiles. These relations determine quotient sets

$$X_A := U/\alpha, \qquad X_B := U/\beta,$$

and hence a canonical factor map

$$\Theta : U \to X_A \times X_B, \qquad \Theta(u) = ([u]_\alpha, [u]_\beta). \tag{2.1.1}$$

The central structural question is therefore the following. *When does the intrinsic comparison structure determine the entire state space through this factor map?* The answer is a single closure principle: rectangular comparison completeness, the chapter-level realization of item (SP3). We prove that this condition is equivalent to bijectivity of Θ, and hence to the existence of a canonical product decomposition

$$U \cong X_A \times X_B$$

realizing the intrinsic congruences as coordinate equalities. The proof proceeds in two complementary layers.

- First, we analyze the comparison predicates directly and show that rectangular completeness is equivalent to bijectivity of the canonical factor map.
- Second, we internalize the same structure inside the Boolean algebra generated by the comparison predicates. The finite-coordinate Boolean algebras form a refinement tower, and the compactness results of chapter 1 allow finite rectangle conditions to be lifted to global splitting statements.

Under the standing principle, and in the chapter-local distinguishability setup

$$\alpha \cap \beta = \Delta_U,$$

these arguments yield equivalent formulations of rectangular completeness in three different languages:

- comparison structure;
- Boolean algebra factorization;
- ultrafilter splitting.

When these conditions hold, the intrinsic symmetry group

$$G := \mathrm{Aut}(U, \mathcal{C})$$

acts diagonally on the canonical product

$$X_A \times X_B,$$

and the quotient data used later in the relational program are determined. This establishes the canonical product decomposition promised in Theorem 0.4.1.

2.2 Primitive comparison worlds

Definition 2.2.1 (Comparison world (recalled))**.** As in Theorem 0.1.1, a *comparison world* is a pair $(U, \mathcal{C})$, where $U \neq \varnothing$ and $\mathcal{C}$ is a family of maps

$$c : U \times U \to \{0, 1\}.$$

In this manuscript such worlds are required to be intrinsic in the sense of Theorem 0.1.1.

Remark 2.2.2. No symmetry, topology, product structure, or background geometry is assumed. Every structure appearing below is extracted from the comparison predicates themselves.

Definition 2.2.3 (Intrinsic symmetry group)**.** Define

$$G := \mathrm{Aut}(U, \mathcal{C}) := \big\{\phi \in \mathrm{Bij}(U) : c(\phi(u), \phi(v)) = c(u, v)\ \forall c \in \mathcal{C},\ \forall u, v \in U\big\}.$$

Proposition 2.2.4. *G is a group under composition.*

Proof. The identity map preserves every predicate, hence lies in G. If $\phi, \psi \in G$, then for every $c \in \mathcal{C}$ and $u, v \in U$,

$$c((\phi \circ \psi)(u), (\phi \circ \psi)(v)) = c(\phi(\psi(u)), \phi(\psi(v))) = c(\psi(u), \psi(v)) = c(u, v),$$

so $\phi \circ \psi \in G$. If $\phi \in G$, fix $c \in \mathcal{C}$ and $u, v \in U$. Write $u = \phi(u')$ and $v = \phi(v')$. Then

$$c(\phi^{-1}(u), \phi^{-1}(v)) = c(u', v') = c(\phi(u'), \phi(v')) = c(u, v).$$

Hence $\phi^{-1} \in G$. ☐

2.3 Intrinsic congruences

Definition 2.3.1 (Left and right profiles)**.** For each $u \in U$, define functions

$$\begin{aligned} \mathsf{L}(u) &: \mathcal{C} \times U \to \{0, 1\}, \\ \mathsf{L}(u)(c, w) &:= c(u, w), \end{aligned}$$

and

$$\begin{aligned} \mathsf{R}(u) &: \mathcal{C} \times U \to \{0, 1\}, \\ \mathsf{R}(u)(c, w) &:= c(w, u). \end{aligned}$$

Definition 2.3.2 (Intrinsic congruences)**.** Define relations α, β on U by

$$u \,\alpha\, v \iff \mathsf{L}(u) = \mathsf{L}(v), \qquad u \,\beta\, v \iff \mathsf{R}(u) = \mathsf{R}(v).$$

Equivalently,

$$u \,\alpha\, v \iff c(u, w) = c(v, w)\ \forall c \in \mathcal{C},\ \forall w \in U,$$

and

$$u \,\beta\, v \iff c(w, u) = c(w, v)\ \forall c \in \mathcal{C},\ \forall w \in U.$$

Lemma 2.3.3. *α and β are equivalence relations.*

Proof. Each is equality of functions in $\{0, 1\}^{\mathcal{C} \times U}$. Reflexivity, symmetry, and transitivity follow immediately. ☐

Lemma 2.3.4 (Maximality among unary comparison invariants)**.** *Let $\sim$ be any equivalence relation on U such that*

$$u \sim v \Longrightarrow c(u,w) = c(v,w) \qquad \forall c \in \mathcal{C},\ \forall w \in U.$$

Then $\sim \subseteq \alpha$. Similarly, any equivalence relation preserving all right predicates is contained in β.

Proof. If $u \sim v$, the stated hypothesis implies $\mathsf{L}(u) = \mathsf{L}(v)$, hence $u\,\alpha\,v$. Therefore every $\sim$-pair is an α-pair. The right-handed statement is identical. □

Lemma 2.3.5 (Invariance)**.** *α and β are G-invariant.*

Proof. Suppose $u\,\alpha\,v$, fix $g \in G$, $c \in \mathcal{C}$, and $w' \in U$. Since g is bijective, there exists $w \in U$ with $w' = g(w)$. Then

$$c(g(u), w') = c(g(u), g(w)) = c(u,w) = c(v,w) = c(g(v), g(w)) = c(g(v), w').$$

Hence $g(u)\,\alpha\,g(v)$. The proof for β is identical. □

2.4 The canonical factor map and its universal property

Define

$$X_A := U/\alpha, \qquad X_B := U/\beta.$$

Definition 2.4.1 (Canonical factor map)**.** The canonical factor map is

$$\Theta : U \to X_A \times X_B, \qquad \Theta(u) := ([u]_\alpha, [u]_\beta).$$

Lemma 2.4.2 (Kernel characterization)**.** *For $u, v \in U$,*

$$\Theta(u) = \Theta(v) \iff u\,\alpha\,v \text{ and } u\,\beta\,v.$$

In particular,

$$\Theta \text{ is injective} \iff \alpha \cap \beta = \Delta_U.$$

Proof. By definition,

$$\Theta(u) = \Theta(v) \iff [u]_\alpha = [v]_\alpha \text{ and } [u]_\beta = [v]_\beta,$$

which is equivalent to $u\,\alpha\,v$ and $u\,\beta\,v$. The injectivity criterion is the special case of equality of points. □

Definition 2.4.3 (Realizing product presentation)**.** A *realizing product presentation* is a bijection

$$p : U \to Y_A \times Y_B$$

such that

$$u \,\alpha\, v \iff \pi_A(p(u)) = \pi_A(p(v)),$$

and

$$u \,\beta\, v \iff \pi_B(p(u)) = \pi_B(p(v)),$$

where π_A, π_B denote the coordinate projections.

Proposition 2.4.4 (Terminality of the canonical factor map)**.** *Let*

$$p : U \to Y_A \times Y_B$$

be a realizing product presentation. Then there exists a unique bijection

$$\Phi : Y_A \times Y_B \to X_A \times X_B$$

such that

$$\Theta = \Phi \circ p.$$

Proof. We define the coordinate maps separately. First define $\phi_A : Y_A \to X_A$ by

$$\phi_A(a) := [p^{-1}(a, b)]_\alpha,$$

where $b \in Y_B$ is arbitrary. We must show that this does not depend on the choice of b. Let $b_1, b_2 \in Y_B$, and set

$$u_i := p^{-1}(a, b_i) \qquad (i = 1, 2).$$

Then

$$\pi_A(p(u_1)) = \pi_A(p(u_2)) = a.$$

Since p realizes α, it follows that $u_1 \,\alpha\, u_2$, hence

$$[p^{-1}(a, b_1)]_\alpha = [p^{-1}(a, b_2)]_\alpha.$$

Thus ϕ_A is well defined. Similarly define $\phi_B : Y_B \to X_B$ by

$$\phi_B(b) := [p^{-1}(a, b)]_\beta,$$

where $a \in Y_A$ is arbitrary. If $a_1, a_2 \in Y_A$, then with $v_i := p^{-1}(a_i, b)$ we have

$$\pi_B(p(v_1)) = \pi_B(p(v_2)) = b.$$

Since p realizes β, we obtain $v_1 \,\beta\, v_2$, so

$$[p^{-1}(a_1, b)]_\beta = [p^{-1}(a_2, b)]_\beta.$$

Hence ϕ_B is well defined. Now define

$$\Phi : Y_A \times Y_B \to X_A \times X_B, \qquad \Phi(a,b) := (\phi_A(a), \phi_B(b)).$$

This is well defined because ϕ_A and ϕ_B are. Let $u \in U$, and write $p(u) = (a,b)$. Then

$$\Phi(p(u)) = (\phi_A(a), \phi_B(b)) = ([u]_\alpha, [u]_\beta) = \Theta(u).$$

Thus

$$\Theta = \Phi \circ p.$$

Uniqueness is immediate from bijectivity of p: if Φ' also satisfies $\Theta = \Phi' \circ p$, then for every $(a,b) \in Y_A \times Y_B$,

$$\Phi(a,b) = \Theta(p^{-1}(a,b)) = \Phi'(a,b).$$

It remains to prove that Φ is bijective. *Injectivity.* Assume

$$\Phi(a_1, b_1) = \Phi(a_2, b_2).$$

Set

$$u_i := p^{-1}(a_i, b_i) \qquad (i = 1, 2).$$

Then

$$\Theta(u_1) = \Phi(p(u_1)) = \Phi(a_1, b_1) = \Phi(a_2, b_2) = \Phi(p(u_2)) = \Theta(u_2).$$

By Theorem 2.4.2,

$$u_1 \, \alpha \, u_2, \qquad u_1 \, \beta \, u_2.$$

Since p realizes α and β, this implies

$$a_1 = \pi_A(p(u_1)) = \pi_A(p(u_2)) = a_2, \qquad b_1 = \pi_B(p(u_1)) = \pi_B(p(u_2)) = b_2.$$

Hence $(a_1, b_1) = (a_2, b_2)$, so Φ is injective. *Surjectivity.* Let $(A, B) \in X_A \times X_B$. Choose $u_A, u_B \in U$ such that

$$[u_A]_\alpha = A, \qquad [u_B]_\beta = B.$$

Write

$$p(u_A) = (a_A, b_A), \qquad p(u_B) = (a_B, b_B).$$

Set

$$u := p^{-1}(a_A, b_B).$$

Then

$$\pi_A(p(u)) = a_A = \pi_A(p(u_A)).$$

Since p realizes α, it follows that $u \, \alpha \, u_A$, hence

$$[u]_\alpha = [u_A]_\alpha = A.$$

Likewise,

$$\pi_B(p(u)) = b_B = \pi_B(p(u_B)),$$

so $u \, \beta \, u_B$, hence

$$[u]_\beta = [u_B]_\beta = B.$$

Therefore

$$\Theta(u) = ([u]_\alpha, [u]_\beta) = (A, B).$$

Since $\Theta = \Phi \circ p$, we obtain

$$(A, B) = \Theta(u) = \Phi(p(u)),$$

and Φ is surjective. Thus Φ is bijective. □

2.5 Rectangular completeness

Definition 2.5.1 (Rectangular completeness)**.** The comparison world $(U, \mathcal{C})$ is *rectangularly complete* if for every $A \in X_A$ and $B \in X_B$ there exists a unique $u \in U$ such that

$$[u]_\alpha = A, \qquad [u]_\beta = B.$$

Equivalently: for every $u, v \in U$ there exists a unique $x \in U$ such that $x \, \alpha \, u$ and $x \, \beta \, v$.

Remark 2.5.2. Rectangular completeness asserts that any left profile can be paired with any right profile, and that this pairing is unique.

2.5.1 A symmetry obstruction

Definition 2.5.3 (Symmetric comparison world)**.** A comparison world $(U, \mathcal{C})$ is *symmetric* if

$$c(u, w) = c(w, u) \qquad \forall c \in \mathcal{C}, \ \forall u, w \in U.$$

Lemma 2.5.4 (Collapse of left and right profiles)**.** *If $(U, \mathcal{C})$ is symmetric, then*

$$\mathsf{L}(u) = \mathsf{R}(u) \qquad \forall u \in U.$$

In particular,

$$\alpha = \beta.$$

Proof. For every $u \in U$, $c \in \mathcal{C}$, and $w \in U$,

$$\mathsf{L}(u)(c, w) = c(u, w) = c(w, u) = \mathsf{R}(u)(c, w).$$

Hence $\mathsf{L}(u) = \mathsf{R}(u)$. Therefore, for $u, v \in U$,

$$u \, \alpha \, v \iff \mathsf{L}(u) = \mathsf{L}(v) \iff \mathsf{R}(u) = \mathsf{R}(v) \iff u \, \beta \, v.$$

□

Proposition 2.5.5 (Symmetry obstructs rectangular completeness)**.** *Assume $|U| > 1$ and $(U, \mathcal{C})$ is symmetric. Then $(U, \mathcal{C})$ is not rectangularly complete. Equivalently,*

$$\Theta : U \to X_A \times X_B$$

is not bijective. More precisely, exactly one of the following occurs:

(K) *If $\alpha \neq \Delta_U$, then Θ is not injective.*

(R) *If $\alpha = \Delta_U$, then Θ is injective but not surjective.*

Proof. By Theorem 2.5.4, $\alpha = \beta$. If $\alpha \neq \Delta_U$, then

$$\alpha \cap \beta = \alpha \neq \Delta_U,$$

so Θ is not injective by Theorem 2.4.2. If $\alpha = \Delta_U$, then also $\beta = \Delta_U$, hence

$$|X_A| = |U|, \qquad |X_B| = |U|.$$

Therefore

$$|X_A \times X_B| = |U|^2 > |U|$$

because $|U| > 1$. Since the image of Θ has cardinality at most $|U|$, Θ cannot be surjective. Thus Θ is never bijective. □

Corollary 2.5.6 (Necessary asymmetry)**.** *Assume $|U| > 1$ and $(U, \mathcal{C})$ is rectangularly complete. Then there exist $c \in \mathcal{C}$ and $u, w \in U$ such that*

$$c(u, w) \neq c(w, u).$$

Proof. This is the contrapositive of Theorem 2.5.5. □

2.5.2 Finite profile support and orbit separation

The next definitions and lemmas record the finite-coordinate facts needed for the intrinsicality consequence without routing through the later Boolean factorization theorem.

Definition 2.5.7 (Generated Boolean algebra)**.** Let B be the Boolean subalgebra of $\mathcal{P}(U)$ generated by the sets

$$L_{c,w} := \{u \in U : \ c(u, w) = 1\}, \qquad R_{c,w} := \{u \in U : \ c(w, u) = 1\},$$

for all $c \in \mathcal{C}$ and $w \in U$. Let $\mathsf{B}_A \subseteq \mathsf{B}$ be the Boolean subalgebra generated by the $L_{c,w}$, and $\mathsf{B}_B \subseteq \mathsf{B}$ the Boolean subalgebra generated by the $R_{c,w}$.

Definition 2.5.8 (Membership equivalence)**.** For a family $\mathcal{F} \subseteq \mathcal{P}(U)$, define a relation $\equiv_{\mathcal{F}}$ on U by

$$u \equiv_{\mathcal{F}} v \iff (\forall E \in \mathcal{F})\, (u \in E \Leftrightarrow v \in E).$$

Lemma 2.5.9 (Classes as membership equivalence)**.** *For $u, v \in U$,*

$$u \,\alpha\, v \iff u \equiv_{\mathsf{B}_A} v, \qquad u \,\beta\, v \iff u \equiv_{\mathsf{B}_B} v.$$

Proof. We prove the first equivalence; the second is identical. Assume $u \,\alpha\, v$. Then by definition,

$$c(u, w) = c(v, w) \qquad \forall c \in \mathcal{C},\ \forall w \in U.$$

Equivalently,

$$u \in L_{c,w} \iff v \in L_{c,w} \qquad \forall c, w.$$

Since B_A is the Boolean algebra generated by the $L_{c,w}$, agreement on all generators implies agreement on every element of B_A. Thus $u \equiv_{\mathsf{B}_A} v$. Conversely, if $u \equiv_{\mathsf{B}_A} v$, then in particular

$$u \in L_{c,w} \iff v \in L_{c,w} \qquad \forall c, w,$$

which means $c(u, w) = c(v, w)$ for all c, w. Hence $u \,\alpha\, v$. □

Definition 2.5.10 (Finite-coordinate subalgebras)**.** Let $\mathcal{C}_0 \subseteq \mathcal{C}$ and $W_0 \subseteq U$ be finite. Define

$$\mathsf{B}_A(\mathcal{C}_0, W_0) := \mathrm{Bool}\big(\{L_{c,w} : (c, w) \in \mathcal{C}_0 \times W_0\}\big) \subseteq \mathsf{B}_A,$$
$$\mathsf{B}_B(\mathcal{C}_0, W_0) := \mathrm{Bool}\big(\{R_{c,w} : (c, w) \in \mathcal{C}_0 \times W_0\}\big) \subseteq \mathsf{B}_B,$$

and

$$\mathsf{B}(\mathcal{C}_0, W_0) := \mathrm{Bool}\big(\mathsf{B}_A(\mathcal{C}_0, W_0) \cup \mathsf{B}_B(\mathcal{C}_0, W_0)\big) \subseteq \mathsf{B}.$$

Lemma 2.5.11 (Finite support)**.** *Every element of B belongs to $\mathsf{B}(\mathcal{C}_0, W_0)$ for some finite $\mathcal{C}_0 \subseteq \mathcal{C}$ and finite $W_0 \subseteq U$. Likewise every element of B_A and of B_B belongs to a corresponding finite-coordinate subalgebra.*

Proof. By definition, B is the Boolean algebra generated by the family $\{L_{c,w}, R_{c,w}\}_{(c,w) \in \mathcal{C} \times U}$. Every element of a generated Boolean algebra is obtained from finitely many generators using finitely many Boolean operations. Thus any given $E \in \mathsf{B}$ depends on only finitely many pairs (c, w). The same argument applies to B_A and B_B. □

Lemma 2.5.12 (Finite orbit separation under intrinsicality)**.** *Let $(U, \mathcal{C})$ be intrinsic, and let $O_1, O_2 \subseteq U$ be distinct G-orbits, where $G = \mathrm{Aut}(U, \mathcal{C})$. If representatives $u \in O_1$ and $v \in O_2$ are distinguished by the comparison data, in the sense that $\mathsf{L}(u) \neq \mathsf{L}(v)$ or $\mathsf{R}(u) \neq \mathsf{R}(v)$, then the distinction is witnessed by a single predicate coordinate. That is, there exist $c \in \mathcal{C}$ and $w \in U$ with*

$$c(u, w) \neq c(v, w)$$

or with

$$c(w, u) \neq c(w, v).$$

Consequently the distinguishing cylinder lies in a finite-coordinate subalgebra $\mathsf{B}(\mathcal{C}_0, W_0)$ with $|\mathcal{C}_0| = |W_0| = 1$, and no comparison-profile separation of orbits requires the completion $\overline{\mathsf{B}}$.

Proof. Suppose first that $\mathsf{L}(u) \neq \mathsf{L}(v)$. By definition, $\mathsf{L}(u)$ and $\mathsf{L}(v)$ are functions

$$\mathcal{C} \times U \to \{0,1\}, \qquad \mathsf{L}(x)(c,w) = c(x,w).$$

Inequality of functions means disagreement at some single point of the domain. Hence there is a pair $(c,w) \in \mathcal{C} \times U$ such that

$$c(u,w) \neq c(v,w).$$

Then one of u, v lies in $L_{c,w}$ and the other does not, and

$$L_{c,w} \in \mathsf{B}(\{c\},\{w\}).$$

The right-profile case is identical: if $\mathsf{R}(u) \neq \mathsf{R}(v)$, then for some (c,w) one has $c(w,u) \neq c(w,v)$, and the cylinder $R_{c,w} \in \mathsf{B}(\{c\},\{w\})$ witnesses the distinction.

Intrinsicality makes this single-coordinate witness internal rather than an external label. If $g \in G$, then

$$c(gu,gw) = c(u,w) \qquad \text{and} \qquad c(gv,gw) = c(v,w),$$

so any left-coordinate distinction is transported equivariantly to the diagonal translate (gu,gv) by the single transported coordinate (c,gw); the right-coordinate case is the same. Thus a comparison distinction cannot first appear as an infinite or completed Boolean combination. If two profiles differ, they differ at one coordinate; if they agree at every coordinate, they are the same profile. □

Remark 2.5.13 (Why no limit-only separation exists). Theorem 2.5.12 is the point at which intrinsicality excludes finitely unwitnessed profile classes. Inequality of profile functions is disagreement at a single coordinate; there is no notion of two profiles that agree at every finite coordinate but differ in the limit, because functions equal at every coordinate are equal. A distinction surviving only in a completion $\overline{\mathsf{B}} \setminus \mathsf{B}$ would require the individual predicate coordinates not to register the distinction while an infinite Boolean combination does. But each predicate is a $\{0,1\}$-valued coordinate, and by intrinsicality each coordinate is transported by the symmetries it helps constitute. The completion adjoins no new comparison separations; it adjoins only infinite Boolean combinations of separations already witnessed coordinatewise.

2.6 Main classification theorem

Theorem 2.6.1 (Exact classification)**.** *The following are equivalent:*

(i) $(U,\mathcal{C})$ *is rectangularly complete.*

(ii) $\Theta : U \to X_A \times X_B$ *is bijective.*

(iii) *There exists a realizing product presentation.*

Moreover, when these conditions hold, the identification

$$U \cong X_A \times X_B$$

is canonical, namely Θ*, and any realizing presentation factors uniquely through* Θ *by Theorem 2.4.4.*

Proof. (i)⇒(ii). Rectangular completeness is exactly the statement that every pair $(A, B) \in X_A \times X_B$ has a unique preimage under Θ. *(ii)⇒(i).* If Θ is bijective, then for every $(A, B) \in X_A \times X_B$ there is a unique

$$u = \Theta^{-1}(A, B)$$

with $[u]_\alpha = A$ and $[u]_\beta = B$. *(ii)⇒(iii).* Take $Y_A = X_A$, $Y_B = X_B$, and $p = \Theta$. *(iii)⇒(ii).* Apply Theorem 2.4.4. If p is a realizing product presentation, then there exists a bijection

$$\Phi : Y_A \times Y_B \to X_A \times X_B$$

with

$$\Theta = \Phi \circ p.$$

Since p is bijective, Θ is bijective. □

Lemma 2.6.2 (Conservative-completion dichotomy)**.** *Let* $(U, \mathcal{C})$ *be intrinsic and locally distinguishable, and let* $(A, B) \in X_A \times X_B$ *be a profile pair with no* $u \in U$ *satisfying* $[u]_\alpha = A$ *and* $[u]_\beta = B$*. Choose representatives* $a \in A$ *and* $b \in B$*, and write the class-determined values*

$$A(c, w) := c(a, w), \qquad B(c, w) := c(w, b),$$

which are independent of the chosen representatives by Theorem 2.3.2. Form the one-point extension

$$U^+ := U \cup \{u_{AB}\},$$

where u_{AB} *is a formal element assigned the comparisons of profile* (A, B)*: for each* $c \in \mathcal{C}$ *and* $w \in U$*, set*

$$c(u_{AB}, w) := A(c, w), \qquad c(w, u_{AB}) := B(c, w),$$

and choose $c(u_{AB}, u_{AB})$ *whenever the same admissibility conditions allow a diagonal value. Then exactly one of the following holds.*

(a) Internal exclusion. *The assignment is inconsistent with the comparison data: some* $c \in \mathcal{C}$*, together with the structural conditions defining admissible profiles in Theorems 2.3.1 and 2.3.2, forbids a state of profile* (A, B)*. Then* (A, B) *is not an admissible profile pair, and no obstruction to surjectivity arises from it.*

(b) Conservative extension. *The assignment is consistent, and the inclusion $U \hookrightarrow U^+$ preserves the finitely supported comparison algebra and automorphism-invariant profile data: restriction from U^+ to U is an isomorphism on every finite-coordinate subalgebra supported in U, and each $g \in \mathrm{Stab}_G(A, B)$ extends to an automorphism of U^+ fixing u_{AB}. Equivalently, the full G-action extends after adjoining the G-orbit of the formal profile pair. In this case the presence or absence of u_{AB} is not registered by any intrinsic predicate, invariant, or finite comparison context supported in U; the two worlds are intrinsically indistinguishable with respect to the pair (A, B).*

There is no third case: an extension that is neither internally excluded nor conservative is impossible.

Proof. The alternatives are mutually exclusive by definition: in (a) no consistent assignment exists, while (b) begins with a consistent assignment. It remains to show exhaustiveness. Suppose (a) fails, so the formal profile assignment is consistent. We prove that (b) holds.

Finite-coordinate algebra is preserved. By Theorem 2.5.11, every element of B has finite support, depending on finitely many pairs (c, w) with $w \in U$. Adjoining u_{AB} introduces evaluations at the new point, but the values $c(u_{AB}, w)$ and $c(w, u_{AB})$ are fixed by the existing profile classes A and B. Hence no finite-coordinate subalgebra supported in U acquires a new value not already dictated by U's comparison data, and restriction from U^+ to U is a Boolean isomorphism on each such finite-coordinate subalgebra.

No new profile distinction is created on old states. Suppose, for contradiction, that adjoining u_{AB} refined a profile class on U. By Theorem 2.5.12, any such separation is witnessed at a single predicate coordinate. If the coordinate is (c, w) with $w \in U$, the disagreement already existed in U, a contradiction. Thus the only possible new coordinate uses $w = u_{AB}$. If $u \,\alpha\, v$ in U, then

$$c(u, u_{AB}) = B(c, u) = c(u, b) = c(v, b) = B(c, v) = c(v, u_{AB}),$$

so the new coordinate does not split an old α-class. Similarly, if $u \,\beta\, v$ in U, then

$$c(u_{AB}, u) = A(c, u) = c(a, u) = c(a, v) = A(c, v) = c(u_{AB}, v),$$

so it does not split an old β-class. Therefore the profile quotients of U^+ restrict to the original profile quotients on U.

Automorphisms extend on the profile stabilizer. Let $g \in \mathrm{Stab}_G(A, B)$. Define g^+ on U^+ by $g^+|_U = g$ and $g^+(u_{AB}) = u_{AB}$. Since g preserves all predicates on U, it remains only to check pairs involving u_{AB}. For $w \in U$, the condition $gA = A$ gives

$$c(g^+u_{AB}, gw) = c(u_{AB}, gw) = A(c, gw) = A(c, w) = c(u_{AB}, w),$$

and $gB = B$ gives

$$c(gw, g^+u_{AB}) = c(gw, u_{AB}) = B(c, gw) = B(c, w) = c(w, u_{AB}).$$

The diagonal value is fixed by consistency of the assigned profile. Thus $g^+ \in \mathrm{Aut}(U^+, \mathcal{C})$. For a general $g \in G$, the same calculation sends u_{AB} to the formal point of profile (gA, gB), so the full action extends after adjoining the orbit of the missing profile pair.

These three facts are precisely the conservative-extension alternative, so if internal exclusion fails, conservative extension holds. Hence the dichotomy is exhaustive. □

Corollary 2.6.3 (Closure forbids conservative omission). *In a world closed under intrinsic description, the conservative-extension case of Theorem 2.6.2 cannot obtain for any unrealized admissible profile pair. For in that case the omission of u_{AB} is maintained by no intrinsic predicate, invariant, or finite comparison context supported in U; by the lemma it is intrinsically undetectable. Such an omission therefore depends on a selection external to the comparison structure. Closure, as expressed by intrinsicality in Theorem 0.1.1 together with the prohibition on external scaffolding in item* (SP1), *admits no externally maintained omission. Hence every admissible profile pair is either internally excluded or already realized; equivalently, no unrealized admissible rectangular cell remains.*

Theorem 2.6.4 (Rectangular completeness under intrinsicality). *Let $(U, \mathcal{C})$ be intrinsic (Theorem 0.1.1) with local distinguishability*

$$\alpha \cap \beta = \Delta_U.$$

Then Θ is bijective; the world is rectangularly complete.

Proof. Injectivity is Theorem 2.4.2 under $\alpha \cap \beta = \Delta_U$. For surjectivity, suppose $(A, B) \in X_A \times X_B$ is an admissible profile pair unrealized in U. By Theorem 2.6.2, the one-point completion at (A, B) is either internally excluded or conservative. Internal exclusion contradicts admissibility. If the completion is conservative, Theorem 2.6.3 applies: a closed world admits no conservative omission, so (A, B) must be realized. Either way no unrealized admissible cell survives. Therefore Θ is surjective, hence bijective. □

Remark 2.6.5. Equivalently, one may realize the missing cells by finite induction. By Theorem 2.6.2, each conservative adjunction leaves the finitely supported profile algebra of Theorem 2.5.11 unchanged and reduces the count of unrealized admissible pairs; by Theorem 2.5.12, no separation requires a completion. The induction and the dichotomy are the same argument viewed from opposite sides: the engine of surjectivity is closure's prohibition on intrinsically undetectable omission, not the bookkeeping of the iteration.

Remark 2.6.6. This does not make rectangular completeness universal. Non-intrinsic worlds, and intrinsic worlds failing $\alpha \cap \beta = \Delta_U$, may fail it; the symmetric worlds of Theorem 2.5.5 remain genuine open systems. The theorem states that an intrinsic, locally distinguishing world is closed: closedness is a consequence of the primitive being a genuine comparison, not an added assumption.

2.7 Failure modes

Definition 2.7.1 (Kernel obstruction)**.** Define

$$K_{\alpha\beta} := \alpha \cap \beta.$$

Proposition 2.7.2 (Injectivity obstruction)**.** *Θ is injective if and only if $K_{\alpha\beta} = \Delta_U$.*

Proof. This is exactly Theorem 2.4.2. □

Definition 2.7.3 (Rectangular deficiency)**.** We say that $(U, \mathcal{C})$ has *rectangular deficiency* if $K_{\alpha\beta} = \Delta_U$ but Θ is not surjective. Equivalently, there exist $A \in X_A$ and $B \in X_B$ such that no $u \in U$ satisfies

$$([u]_\alpha, [u]_\beta) = (A, B).$$

Theorem 2.7.4 (Exhaustive trichotomy)**.** *Exactly one of the following holds:*

(D) *Θ is bijective.*

(K) *$K_{\alpha\beta} \neq \Delta_U$.*

(R) *$K_{\alpha\beta} = \Delta_U$ but Θ is not surjective.*

Proof. Either Θ is bijective or it is not. If it is not bijective, then either injectivity fails or surjectivity fails. By Theorem 2.7.2, injectivity fails exactly when $K_{\alpha\beta} \neq \Delta_U$. If injectivity holds but surjectivity fails, then by definition we are in the rectangle-deficient case. The three cases are mutually exclusive and exhaustive. □

Proposition 2.7.5 (Minimal counterexamples)**.** *There exist comparison worlds realizing each obstruction:*

1. *a kernel-obstructed example with $|U| = 2$;*
2. *a rectangle-deficient example with $|U| = 3$ and $K_{\alpha\beta} = \Delta_U$.*

Proof. For kernel obstruction, let $U = \{a, b\}$ and let $\mathcal{C} = \{c\}$ with $c(u, v) = 0$ for all pairs. Then all left and right profiles are identical, so

$$\alpha = \beta = U \times U, \qquad K_{\alpha\beta} \neq \Delta_U.$$

For rectangle deficiency, let $U = \{x, y, z\}$ and let $\mathcal{C} = \{c\}$, where $c(u, v) = 1$ if and only if $u = v$. Then both left and right profiles separate points, so

$$\alpha = \beta = \Delta_U, \qquad K_{\alpha\beta} = \Delta_U.$$

Hence $|X_A| = |X_B| = 3$, while $|U| = 3$. Therefore

$$|X_A \times X_B| = 9 > |U|,$$

so Θ cannot be surjective. □

2.8 Canonical diagonal symmetry once decomposable

Assume from now on that the equivalent conditions of Theorem 2.6.1 hold, so that Θ is bijective.

Definition 2.8.1 (Induced factor actions)**.** For $g \in G$ define actions on X_A and X_B by

$$g \cdot [u]_\alpha := [g(u)]_\alpha, \qquad g \cdot [u]_\beta := [g(u)]_\beta.$$

Lemma 2.8.2 (Well-definedness)**.** *The actions in Theorem 2.8.1 are well defined.*

Proof. If $[u]_\alpha = [v]_\alpha$, then $u \,\alpha\, v$. By Theorem 2.3.5, $g(u) \,\alpha\, g(v)$, hence $[g(u)]_\alpha = [g(v)]_\alpha$. The proof for X_B is identical. □

Theorem 2.8.3 (Diagonal action theorem)**.** *Under the identification $U \cong X_A \times X_B$ via Θ, the group $G = \mathrm{Aut}(U, \mathcal{C})$ acts diagonally:*

$$\Theta(g(u)) = g \cdot \Theta(u) \qquad \forall g \in G,\ \forall u \in U,$$

that is,

$$g \cdot (A, B) := (g \cdot A,\ g \cdot B).$$

Proof. By definition,

$$\Theta(g(u)) = ([g(u)]_\alpha, [g(u)]_\beta) = (g \cdot [u]_\alpha,\ g \cdot [u]_\beta) = g \cdot \Theta(u).$$

□

Corollary 2.8.4 (Canonical diagonal redundancy data)**.** *Under rectangular completeness, the canonical data*

$$X_A := U/\alpha, \qquad X_B := U/\beta, \qquad G := \mathrm{Aut}(U, \mathcal{C})$$

determine a canonical diagonal action on

$$X := X_A \times X_B$$

and hence a canonical quotient

$$\mathsf{Phys} := X/G.$$

Proof. This is immediate from Theorems 2.6.1 and 2.8.3. □

2.9 Boolean algebra internalization

We now reformulate the structure of Theorems 2.6.1, 2.8.3 and 2.8.4 inside the Boolean algebra generated by the primitive comparison predicates.

2.9.1 The comparison Boolean algebra

The comparison Boolean algebra, membership equivalence, finite-coordinate subalgebras, and finite-support lemmas were introduced in section 2.5.2. We use those definitions throughout the Boolean internalization below.

Remark 2.9.1. All atomic arguments below are carried out inside finite-coordinate subalgebras. No global atom of the full algebra B is invoked.

2.9.2 Finite-predicate refinement tower and compactness

We now make explicit the refinement tower to which chapter 1 applies. This is the point at which the finite Boolean structure of the comparison predicates is connected to the global admissibility mechanism isolated in chapter 1. Fix an increasing exhaustion

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq \mathcal{C} \times U, \qquad \bigcup_{n\in\mathbb{N}} F_n = \mathcal{C} \times U.$$

Define

$$\mathsf{B}^{(n)} := \mathrm{Bool}\big(\{L_{c,w}, R_{c,w} : (c,w) \in F_n\}\big) \subseteq \mathcal{P}(U),$$

$$\mathsf{B}_A^{(n)} := \mathrm{Bool}\big(\{L_{c,w} : (c,w) \in F_n\}\big), \qquad \mathsf{B}_B^{(n)} := \mathrm{Bool}\big(\{R_{c,w} : (c,w) \in F_n\}\big).$$

Then

$$\mathsf{B}^{(n)} \subseteq \mathsf{B}^{(n+1)}, \qquad \mathsf{B}_A^{(n)} \subseteq \mathsf{B}_A^{(n+1)}, \qquad \mathsf{B}_B^{(n)} \subseteq \mathsf{B}_B^{(n+1)},$$

and

$$\mathsf{B} = \bigcup_{n\in\mathbb{N}} \mathsf{B}^{(n)}, \qquad \mathsf{B}_A = \bigcup_{n\in\mathbb{N}} \mathsf{B}_A^{(n)}, \qquad \mathsf{B}_B = \bigcup_{n\in\mathbb{N}} \mathsf{B}_B^{(n)}.$$

For each n, let $\mathrm{At}(\mathsf{B}_A^{(n)})$ and $\mathrm{At}(\mathsf{B}_B^{(n)})$ denote the atoms of these finite Boolean algebras. Define $\mathsf{J}^{(n)} \subseteq \mathsf{B}^{(n)}$ to be the ideal generated by the empty rectangles

$$A \cap B, \qquad A \in \mathrm{At}(\mathsf{B}_A^{(n)}), \quad B \in \mathrm{At}(\mathsf{B}_B^{(n)}), \quad A \cap B = \varnothing.$$

Lemma 2.9.2 (Compatibility of the finite rectangle ideals)**.** *For every n,*

$$\mathsf{J}^{(n)} = \mathsf{J}^{(n+1)} \cap \mathsf{B}^{(n)}.$$

Proof. Every atom of $\mathsf{B}_A^{(n+1)}$ is contained in a unique atom of $\mathsf{B}_A^{(n)}$, and similarly for $\mathsf{B}_B^{(n+1)}$ and $\mathsf{B}_B^{(n)}$. Thus an empty rectangle at stage $n+1$ restricts to an empty rectangle at stage n. Conversely, an empty rectangle at stage n is the finite union of all stage-$(n+1)$ rectangles lying over it, and each of those is empty. The generated ideals therefore agree under restriction. □

Set

$$\mathsf{J}_\infty := \left\langle \bigcup_{n\in\mathbb{N}} \mathsf{J}^{(n)} \right\rangle_{\mathsf{B}}.$$

By Theorems 1.6.4, 1.7.2 and 1.8.1, the tower

$$\big(U, (\mathsf{B}^{(n)})_{n\in\mathbb{N}}, (\mathsf{J}^{(n)})_{n\in\mathbb{N}}\big)$$

admits a global admissible ultrafilter exactly when no finite family of excluded finite-coordinate rectangles covers U, and admissible global ultrafilters are exactly coherent admissible towers of stage ultrafilters:

$$\mathrm{UF}(\mathsf{B}; \mathsf{J}_\infty) \cong \varprojlim_n \mathrm{UF}(\mathsf{B}^{(n)}; \mathsf{J}^{(n)}).$$

This is the compactness input used in the reverse implication of Theorem 2.9.5.

Theorem 2.9.3 (Compactness from intrinsicality)**.** *For an intrinsic comparison world (Theorem 0.1.1), the finite-coordinate refinement tower has limit equal to the direct union*

$$\mathsf{B} = \bigcup_{n\in\mathbb{N}} \mathsf{B}^{(n)},$$

and every comparison profile class separated by intrinsic comparison data is decided at a finite stage. Consequently the finite-to-global boundary holds without separate hypothesis: global admissibility fails if and only if finitely many stage-exclusions cover U, as in Theorem 1.6.4.

Proof. The equality $\mathsf{B} = \bigcup_n \mathsf{B}^{(n)}$ is exactly the finite-support content of Theorem 2.5.11 applied to the chosen exhaustion of $\mathcal{C} \times U$. Intrinsicality makes the profile information internal to the symmetry structure: each predicate is preserved by every element of $G = \mathrm{Aut}(U, \mathcal{C})$, and the family of left and right cylinders is transported within the comparison-generated Boolean algebra.

Suppose a comparison profile class required a completion element outside the finite union tower. Then it would determine a profile distinction not witnessed by any finite-coordinate subalgebra. This is impossible by Theorem 2.5.12: any distinction between left or right profile functions is witnessed by a single predicate coordinate. That coordinate lies in a finite-coordinate subalgebra, and Theorem 2.5.11 then places every Boolean combination of such witnesses in some $\mathsf{B}^{(n)}$. Thus no profile class is finitely unwitnessed, the limit requires no additional completion stage, and the admissibility boundary is precisely the quantifier-reversal boundary of Theorem 1.6.4. □

2.9.3 Rectangular completeness as a Boolean factorization property

From this point onward in the Boolean part of the chapter, under the comparison-completeness clause item (SP3), we work in the local distinguishability setup

$$\alpha \cap \beta = \Delta_U.$$

Definition 2.9.4 (Rectangular factorization property)**.** We say that B has the *rectangular factorization property* if:

(F1) for every nonempty $A \in \mathsf{B}_A$ and nonempty $B \in \mathsf{B}_B$,

$$A \cap B \neq \varnothing;$$

(F2) every $E \in \mathsf{B}$ is a finite union of sets of the form $A \cap B$, with $A \in \mathsf{B}_A$ and $B \in \mathsf{B}_B$.

Theorem 2.9.5 (Boolean internalization of rectangular completeness)**.** *In the local distinguishability setup*

$$\alpha \cap \beta = \Delta_U.$$

Then rectangular completeness holds if and only if B *satisfies the rectangular factorization property.*

Proof. (⇒). Assume rectangular completeness. By Theorem 2.6.1, $\Theta : U \to X_A \times X_B$ is bijective. Identify U with $X_A \times X_B$ through Θ. If $A \in \mathsf{B}_A$, then by Theorem 2.5.9 membership in A depends only on the α-class, hence only on the first coordinate. Therefore there exists a subset $A_A \subseteq X_A$ such that

$$A = A_A \times X_B.$$

Similarly, every $B \in \mathsf{B}_B$ has the form

$$B = X_A \times B_B$$

for some $B_B \subseteq X_B$. Now if $A \in \mathsf{B}_A$ and $B \in \mathsf{B}_B$ are nonempty, then

$$A = A_A \times X_B, \qquad B = X_A \times B_B$$

with $A_A \neq \varnothing$ and $B_B \neq \varnothing$, so

$$A \cap B = (A_A \times X_B) \cap (X_A \times B_B) = A_A \times B_B \neq \varnothing.$$

Thus *(F1)* holds. Moreover, B is generated by the left and right coordinate cylinders, and every finite Boolean combination of such cylinders is a finite union of rectangles $A_A \times B_B$. Transporting back through Θ yields *(F2)*. *(⇐).* Assume *(F1)* and *(F2)*.

We first prove surjectivity of Θ. Fix $A \in X_A$ and $B \in X_B$, and choose representatives $u_A, u_B \in U$ such that

$$[u_A]_\alpha = A, \qquad [u_B]_\beta = B.$$

For each n, let $A_n \in \mathrm{At}(\mathsf{B}_A^{(n)})$ be the unique atom containing u_A, and let $B_n \in \mathrm{At}(\mathsf{B}_B^{(n)})$ be the unique atom containing u_B. Since both are nonempty, *(F1)* implies

$$A_n \cap B_n \neq \varnothing \qquad \text{for every } n.$$

Choose $x_n \in A_n \cap B_n$, and let $\mathfrak{u}_n$ be the principal ultrafilter of $\mathsf{B}^{(n)}$ determined by x_n. Because $x_n \in A_n \cap B_n$ and the ideal $\mathsf{J}^{(n)}$ is generated by empty rectangles, one has

$$\mathfrak{u}_n \in \mathrm{UF}(\mathsf{B}^{(n)}; \mathsf{J}^{(n)}).$$

Moreover, the atoms $A_{n+1} \subseteq A_n$ and $B_{n+1} \subseteq B_n$ refine coherently, so the stage ultrafilters may be chosen coherently. By Theorem 1.7.2, there exists

$$\mathfrak{u} \in \mathrm{UF}(\mathsf{B}; \mathsf{J}_\infty)$$

whose restriction to $\mathsf{B}^{(n)}$ is $\mathfrak{u}_n$ for every n. By construction, $\mathfrak{u}$ contains every set $E \in \mathsf{B}_A$ containing u_A and every set $F \in \mathsf{B}_B$ containing u_B. Hence the complete left-membership data encoded by $\mathfrak{u}$ agree with those of u_A, and the complete right-membership data agree with those of u_B. Equivalently, $\mathfrak{u}$ determines a state $u \in U$ satisfying

$$[u]_\alpha = [u_A]_\alpha = A, \qquad [u]_\beta = [u_B]_\beta = B.$$

Thus Θ is surjective. We now prove injectivity. Since

$$\alpha \cap \beta = \Delta_U,$$

injectivity of Θ is immediate from Theorem 2.4.2. Therefore Θ is bijective. By Theorem 2.6.1, rectangular completeness follows. ☐

2.9.4 Rectangle algebras and ultrafilter splitting

We now reformulate the same condition in algebraic and ultrafilter language.

Definition 2.9.6 (Ultrafilter)**.** Let $\mathcal{A}$ be a Boolean algebra. An *ultrafilter* on $\mathcal{A}$ is a subset $\mathfrak{u} \subseteq \mathcal{A}$ such that:

1. $1 \in \mathfrak{u}$ and $0 \notin \mathfrak{u}$;
2. $a, b \in \mathfrak{u} \Rightarrow a \wedge b \in \mathfrak{u}$;
3. $a \in \mathfrak{u}$ and $a \leq b \Rightarrow b \in \mathfrak{u}$;
4. for each $a \in \mathcal{A}$, exactly one of a and $\neg a$ lies in $\mathfrak{u}$.

Definition 2.9.7 (Marginals)**.** For $\mathfrak{u} \in \mathrm{UF}(\mathsf{B})$, define

$$\mathfrak{u}_A := \mathfrak{u} \cap \mathsf{B}_A \in \mathrm{UF}(\mathsf{B}_A), \qquad \mathfrak{u}_B := \mathfrak{u} \cap \mathsf{B}_B \in \mathrm{UF}(\mathsf{B}_B).$$

Lemma 2.9.8. *If* $\mathfrak{u} \in \mathrm{UF}(\mathsf{B})$*, then* $\mathfrak{u}_A \in \mathrm{UF}(\mathsf{B}_A)$ *and* $\mathfrak{u}_B \in \mathrm{UF}(\mathsf{B}_B)$*.*

Proof. Restriction of an ultrafilter to a Boolean subalgebra preserves all ultrafilter axioms. ☐

Definition 2.9.9 (Rectangle algebra)**.** Let $\mathcal{A}, \mathcal{D}$ be Boolean algebras. A *rectangle algebra for* $(\mathcal{A}, \mathcal{D})$ is a Boolean algebra $\mathcal{T}$ equipped with homomorphisms

$$\iota_A : \mathcal{A} \to \mathcal{T}, \qquad \iota_D : \mathcal{D} \to \mathcal{T}$$

such that:

1. the images $\iota_A(\mathcal{A}) \cup \iota_D(\mathcal{D})$ generate $\mathcal{T}$;
2. for any Boolean algebra $\mathcal{E}$ and homomorphisms $f : \mathcal{A} \to \mathcal{E}$, $g : \mathcal{D} \to \mathcal{E}$, there exists a unique homomorphism $h : \mathcal{T} \to \mathcal{E}$ such that

$$h \circ \iota_A = f, \qquad h \circ \iota_D = g.$$

When such an algebra exists, we write

$$\mathcal{T} = \mathcal{A} \otimes \mathcal{D}.$$

Proposition 2.9.10 (Existence inside $\mathcal{P}(U)$)**.** *Let $\mathcal{T}$ be the Boolean subalgebra of $\mathcal{P}(U)$ generated by all intersections $A \cap B$ with $A \in \mathsf{B}_A$ and $B \in \mathsf{B}_B$. Then $\mathcal{T}$ is a rectangle algebra for $(\mathsf{B}_A, \mathsf{B}_B)$.*

Proof. Set

$$\mathcal{T} := \mathrm{Bool}\big(\{A \cap B : \ A \in \mathsf{B}_A, \ B \in \mathsf{B}_B\}\big) \subseteq \mathcal{P}(U).$$

Define

$$\iota_A(A) := A, \qquad \iota_D(B) := B.$$

The images generate $\mathcal{T}$ by construction. Now let $f : \mathsf{B}_A \to \mathcal{E}$ and $g : \mathsf{B}_B \to \mathcal{E}$ be Boolean homomorphisms into a Boolean algebra $\mathcal{E}$. On rectangle generators define

$$h(A \cap B) := f(A) \wedge g(B).$$

Because $\mathcal{T}$ is generated by these intersections and Boolean operations, h extends uniquely to a homomorphism

$$h : \mathcal{T} \to \mathcal{E}.$$

This gives the universal property. □

Theorem 2.9.11 (Ultrafilter splitting characterization)**.** *In the local distinguishability setup*

$$\alpha \cap \beta = \Delta_U.$$

The following are equivalent:

(a) *Rectangular completeness holds.*

(b) B *satisfies the rectangular factorization property.*

(c) $\mathsf{B} \cong \mathsf{B}_A \otimes \mathsf{B}_B$.

(d) *The marginal map*

$$\Sigma : \mathrm{UF}(\mathsf{B}) \to \mathrm{UF}(\mathsf{B}_A) \times \mathrm{UF}(\mathsf{B}_B), \qquad \Sigma(\mathfrak{u}) := (\mathfrak{u}_A, \mathfrak{u}_B),$$

is bijective.

Proof. (a)⇔(b). This is Theorem 2.9.5. *(b)⇒(c).* Let $\mathcal{T}$ be as in Theorem 2.9.10. By *(F2)*, every $E \in \mathsf{B}$ is a finite union of intersections $A \cap B$, so

$$\mathsf{B} \subseteq \mathcal{T}.$$

Conversely, every generator $A \cap B$ of $\mathcal{T}$ already belongs to B, so $\mathcal{T} \subseteq \mathsf{B}$. Hence

$$\mathsf{B} = \mathcal{T}.$$

Therefore B satisfies the universal property of $\mathsf{B}_A \otimes \mathsf{B}_B$. *(c)⇒(d).* Assume $\mathsf{B} = \mathsf{B}_A \otimes \mathsf{B}_B$, via a fixed identification. First, Σ is well defined by Theorem 2.9.8. *Injectivity.* Suppose

$$\Sigma(\mathfrak{u}) = \Sigma(\mathfrak{v}).$$

Since B is generated by the rectangle generators

$$A \otimes B, \qquad A \in \mathsf{B}_A,\ B \in \mathsf{B}_B,$$

it suffices to show agreement on these generators. For any ultrafilter $\mathfrak{u}$,

$$A \otimes B \in \mathfrak{u} \iff A \in \mathfrak{u}_A \text{ and } B \in \mathfrak{u}_B.$$

The right-hand side depends only on $\Sigma(\mathfrak{u})$. Thus $\mathfrak{u}$ and $\mathfrak{v}$ agree on all generators, hence on all of B. So Σ is injective. *Surjectivity.* Let

$$(\mathfrak{a}, \mathfrak{b}) \in \mathrm{UF}(\mathsf{B}_A) \times \mathrm{UF}(\mathsf{B}_B).$$

Consider the family

$$\mathcal{G}(\mathfrak{a}, \mathfrak{b}) := \{A \cap B :\ A \in \mathfrak{a},\ B \in \mathfrak{b}\} \subseteq \mathsf{B}.$$

We claim that $\mathcal{G}(\mathfrak{a}, \mathfrak{b})$ has the finite intersection property. Let

$$A_1 \cap B_1, \dots, A_n \cap B_n \in \mathcal{G}(\mathfrak{a}, \mathfrak{b}).$$

Then

$$A^* := \bigcap_{i=1}^{n} A_i \in \mathfrak{a}, \qquad B^* := \bigcap_{i=1}^{n} B_i \in \mathfrak{b}.$$

Since ultrafilters do not contain 0, both A^* and B^* are nonempty. By *(F1)*,

$$A^* \cap B^* \neq \varnothing.$$

Hence

$$\bigcap_{i=1}^{n}(A_i \cap B_i) = A^* \cap B^* \neq \varnothing.$$

So $\mathcal{G}(\mathfrak{a}, \mathfrak{b})$ has the finite intersection property. By the ultrafilter extension theorem, it extends to some $\mathfrak{u} \in \mathrm{UF}(\mathsf{B})$. By construction,

$$\mathfrak{u}_A = \mathfrak{a}, \qquad \mathfrak{u}_B = \mathfrak{b}.$$

Thus Σ is surjective. *(d)⇒(b).* Assume Σ is bijective. To prove *(F1)*, let $A \in \mathsf{B}_A$ and $B \in \mathsf{B}_B$ be nonempty. Choose ultrafilters $\mathfrak{a} \in \mathrm{UF}(\mathsf{B}_A)$ and $\mathfrak{b} \in \mathrm{UF}(\mathsf{B}_B)$ such that

$$A \in \mathfrak{a}, \qquad B \in \mathfrak{b}.$$

By surjectivity of Σ, there exists $\mathfrak{u} \in \mathrm{UF}(\mathsf{B})$ with

$$\Sigma(\mathfrak{u}) = (\mathfrak{a}, \mathfrak{b}).$$

Then $A, B \in \mathfrak{u}$, hence

$$A \cap B \in \mathfrak{u}.$$

Since an ultrafilter cannot contain $\varnothing$, it follows that

$$A \cap B \neq \varnothing.$$

Thus *(F1)* holds. To prove *(F2)*, let

$$\mathsf{B}_{\mathrm{rect}} := \mathrm{Bool}\big(\{A \cap B : \ A \in \mathsf{B}_A, \ B \in \mathsf{B}_B\}\big) \subseteq \mathsf{B}.$$

Assume for contradiction that

$$\mathsf{B}_{\mathrm{rect}} \neq \mathsf{B}.$$

Then there exists $E \in \mathsf{B} \setminus \mathsf{B}_{\mathrm{rect}}$. Since $\mathsf{B}_{\mathrm{rect}}$ is a proper Boolean subalgebra of B, there exist distinct ultrafilters $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathrm{UF}(\mathsf{B})$ which agree on $\mathsf{B}_{\mathrm{rect}}$ but disagree on E. In particular they agree on B_A and on B_B, since both subalgebras lie in $\mathsf{B}_{\mathrm{rect}}$. Therefore

$$\Sigma(\mathfrak{u}_1) = \Sigma(\mathfrak{u}_2),$$

contradicting injectivity of Σ. Hence

$$\mathsf{B} = \mathsf{B}_{\mathrm{rect}},$$

which is exactly *(F2)*. □

Remark 2.9.12. No appeal to Stone duality is required in the proof of Theorem 2.9.11. The only external inputs are the ultrafilter extension theorem and the compactness results of chapter 1.

Theorem 2.9.13 (Rigidity of comparison completeness)**.** *Under the standing principle Standing Principle 1, let* $(U, \mathcal{C})$ *be a comparison world in the local distinguishability setup*

$$\alpha \cap \beta = \Delta_U.$$

Then the following are equivalent:

(1) $(U, \mathcal{C})$ *is rectangularly complete.*

(2) *The canonical factor map*

$$\Theta : U \to X_A \times X_B$$

is bijective.

(3) *There exists a product presentation*

$$U \cong Y_A \times Y_B$$

whose coordinate congruences coincide with α *and* β*.*

(4) *The comparison Boolean algebra satisfies the rectangular factorization property.*

(5) *The comparison Boolean algebra decomposes as*

$$\mathsf{B} \cong \mathsf{B}_A \otimes \mathsf{B}_B.$$

(6) *The marginal ultrafilter map*

$$\Sigma : \mathrm{UF}(\mathsf{B}) \to \mathrm{UF}(\mathsf{B}_A) \times \mathrm{UF}(\mathsf{B}_B), \qquad \Sigma(\mathfrak{u}) := (\mathfrak{u}_A, \mathfrak{u}_B),$$

is bijective.

When these conditions hold, the product decomposition

$$U \cong X_A \times X_B$$

is canonical, every realizing product presentation factors uniquely through Θ*, the symmetry group* $G = \mathrm{Aut}(U, \mathcal{C})$ *acts diagonally on* $X_A \times X_B$*, and the quotient data*

$$\mathsf{Phys} = (X_A \times X_B)/G$$

are determined.

Proof. The equivalence of (1), (2), and (3) is Theorem 2.6.1. The equivalence of (1) and (4) is Theorem 2.9.5. The equivalence of (1), (4), (5), and (6) is Theorem 2.9.11. The uniqueness and rigidity statements follow from Theorems 2.4.4, 2.8.3 and 2.8.4. □

2.10 Structural completion of comparison worlds

Starting from a comparison world $(U, \mathcal{C})$, this chapter has identified the intrinsic congruences α and β, the canonical quotient sets

$$X_A := U/\alpha, \qquad X_B := U/\beta,$$

and the factor map

$$\Theta : U \to X_A \times X_B, \qquad \Theta(u) = ([u]_\alpha, [u]_\beta).$$

Building on the compactness boundary from Chapter 1, the chapter has shown that local intrinsic distinguishability together with rectangular comparison completeness rigidly determines the global comparison structure.

Theorem 2.10.1 (Structural completion of comparison worlds)**.** *Under the standing principle Standing Principle 1, let $(U, \mathcal{C})$ be a comparison world satisfying the compactness boundary condition of item* (SP2)*, the local distinguishability condition*

$$\alpha \cap \beta = \Delta_U,$$

and the closure condition of item (SP3)*, equivalently rectangular comparison completeness. Then the following structure is canonically determined.*

(i) *The canonical factor map*

$$\Theta : U \to X_A \times X_B$$

is bijective. Hence

$$U \cong X_A \times X_B.$$

(ii) *The intrinsic symmetry group*

$$G := \mathrm{Aut}(U, \mathcal{C})$$

acts diagonally on

$$X := X_A \times X_B.$$

(iii) *The comparison world therefore determines the orbit projection*

$$\pi : X \to \mathsf{Phys} := X/G.$$

Proof. Item (i) is the comparison completeness classification theorem (Theorem 2.6.1). Item (ii) is the diagonal action theorem (Theorem 2.8.3). Item (iii) follows from the canonical orbit construction (Theorem 2.8.4). □

The same closure condition has already been expressed in three equivalent Chapter 2 languages by Theorem 2.9.13: comparison-theoretic, Boolean-algebraic, and ultrafilter-theoretic. Thus the canonical product decomposition emerges simultaneously as relational rigidity, Boolean factorization, and ultrafilter splitting, without introducing any semantic content beyond the orbit quotient already fixed by the comparison data.

Remark 2.10.2 (Interpretation). The theorem shows that once Boolean compactness, intrinsic distinguishability, and rectangular comparison completeness hold, the global structure of the comparison world is rigid. The state space admits a canonical product decomposition, the intrinsic symmetry group acts diagonally on that product, and the orbit quotient Phys is thereby fixed by the comparison data themselves. Quotient semantics is the next chapter's semantic reading of this determined quotient backbone, not an additional structure imposed from outside.

Remark 2.10.3 (Role in the manuscript). Theorem 2.10.1 closes Part I's comparison stage by fixing the canonical product, diagonal symmetry, and orbit quotient data, together with their equivalent Boolean and ultrafilter formulations already proved in this chapter. Chapter 3 then starts from this quotient backbone to establish closed-system quotient semantics and the subsystem-attribution boundary.

2.11 Conclusion

Part I closes with a rigidity theorem of exact scope: rectangular comparison completeness determines canonical product structure, diagonal symmetry, the canonical orbit quotient, and the matching Boolean and ultrafilter formulations without residual structural freedom. Chapter 3 starts from this fixed quotient backbone and establishes quotient descent before classifying the remaining admissible non-quotient enrichment loci.

Part II

Quotient Semantics and the No-Go Theorem

Chapter 3

Diagonal Redundancy and the Obstruction to Subsystem Attribution

3.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the quotient-descent and transport-visibility clauses (items (SP4) and (SP5)). Taking the canonical product data of chapter 2 as input, it establishes quotient descent and the subsystem-attribution boundary used in chapters 4 and 5. Chapter 2 established that rectangular comparison completeness determines a canonical product presentation

$$X := X_A \times X_B, \qquad X_A := U/\alpha, \qquad X_B := U/\beta,$$

together with a canonical diagonal action of the intrinsic symmetry group

$$G := \mathrm{Aut}(U, \mathcal{C})$$

on X; see Theorems 2.8.3 and 2.9.13. Accordingly, the primitive comparison world determines canonically the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G,$$

which is exactly the quotient data produced in Theorem 2.8.4. At this stage the remaining question is semantic. Once the comparison world has canonically collapsed to the diagonal-redundancy data

$$X = X_A \times X_B, \qquad G \curvearrowright X, \qquad \pi : X \to \mathsf{Phys},$$

it must be determined which state reports survive as intrinsic content of the closed system, and where any further non-quotient structure could possibly enter. The first point is descent. Once the diagonal action is fixed, the natural admissibility criterion for closed-system state reports is orbit invariance: admissible reports are exactly the

G-invariant maps on X, equivalently the maps that factor uniquely through the orbit projection

$$\pi : X \to \mathsf{Phys}.$$

Theorem 3.2.4 shows that this factorization is automatic. Closed-system semantics is therefore determined to be quotient semantics. The second point is an obstruction theorem. Within quotient semantics, pure orbit motion carries no invariant subsystem attribution. Every coherent report is constant along a pure diagonal orbit loop, and any two pointwise orbit-equivalent supported representatives of such a loop have identical coherent report values. In particular, the distinction

which subsystem moved?

does not define a quotient-level invariant of the closed system. The third point is structural classification. Finite comparison protocols may extend quotient semantics, but within the extension formalism introduced below the possibilities are exhausted: if such an extension is not endpoint-determined, then additional structure enters through at least one of two classified loci, and possibly both:

(i) representative selection, that is, a section of π;

(ii) morphism-level enrichment, equivalently route-dependent transport or nontrivial loop defect.

No third locus exists within that formalism. The chapter therefore identifies the precise obstruction boundary of closed quotient semantics. As shown below, every coherent report factors uniquely through the orbit projection. Consequently, any invariant content expressible by coherent reports is orbit-level content in Phys. Within the extension formalism introduced below, any content beyond that must arise through at least one of the two enrichment mechanisms above, and possibly through both together.

3.2 Closed-system quotient semantics

Throughout this chapter, admissibility is read through item (SP4): semantic content is required to descend through intrinsic quotient maps, while non-quotient residue appears only through the transport mechanisms analyzed later (item (SP5)). Throughout the chapter, G acts diagonally on

$$X = X_A \times X_B, \qquad g \cdot (x_A, x_B) := (g \cdot x_A, g \cdot x_B),$$

and

$$\pi : X \to \mathsf{Phys} := X/G$$

denotes the orbit projection.

Definition 3.2.1 (Coherent report)**.** Let S be a set. A map

$$R : X \to S$$

is *coherent* if

$$R(g \cdot x) = R(x) \qquad \text{for all } g \in G,\ x \in X.$$

Equivalently, R is constant on G-orbits.

Because diagonal redundancy identifies orbit-equivalent representatives, orbit invariance is not an additional convention but the intrinsic admissibility criterion for closed-system state reports.

Definition 3.2.2 (Closed-system semantics)**.** A *closed-system semantics* on X declares the admissible state reports to be exactly the coherent reports.

Definition 3.2.3 (Orbit quotient)**.** The *orbit quotient* of the diagonal action is

$$\mathsf{Phys} := X/G,$$

equipped with the canonical projection

$$\pi : X \to \mathsf{Phys}, \qquad \pi(x) := [x].$$

Theorem 3.2.4 (Canonical descent)**.** *Let $R : X \to S$ be coherent. Then there exists a unique map*

$$\widetilde{R} : \mathsf{Phys} \to S$$

such that

$$R = \widetilde{R} \circ \pi.$$

Proof. Define

$$\widetilde{R}([x]) := R(x).$$

To prove that this is well defined, suppose $[x] = [y]$ in Phys. Then $y = g \cdot x$ for some $g \in G$. Since R is coherent,

$$R(y) = R(g \cdot x) = R(x).$$

Hence $\widetilde{R}([x])$ is independent of the chosen representative. Now for every $x \in X$,

$$(\widetilde{R} \circ \pi)(x) = \widetilde{R}([x]) = R(x),$$

so $R = \widetilde{R} \circ \pi$. For uniqueness, let $f : \mathsf{Phys} \to S$ also satisfy $R = f \circ \pi$. Then for any $[x] \in \mathsf{Phys}$,

$$f([x]) = f(\pi(x)) = R(x) = \widetilde{R}([x]).$$

Therefore $f = \widetilde{R}$. □

Proposition 3.2.5 (Universal property of the orbit quotient)**.** *For every set S and every coherent map*

$$R : X \to S,$$

there exists a unique map

$$\widetilde{R} : \mathsf{Phys} \to S$$

such that

$$R = \widetilde{R} \circ \pi.$$

Equivalently, $\pi : X \to \mathsf{Phys}$ is universal among coherent maps out of X.

Proof. This is exactly Theorem 3.2.4. □

Corollary 3.2.6 (Coherence $\iff$ descent)**.** *For a map $R : X \to S$, the following are equivalent.*

(i) *R is coherent;*

(ii) *R is constant on G-orbits;*

(iii) *there exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ such that*

$$R = \widetilde{R} \circ \pi.$$

Proof. (i)⇒(iii). This is Theorem 3.2.4. *(iii)⇒(ii).* If $R = \widetilde{R} \circ \pi$, then for $x, y \in X$ with $\pi(x) = \pi(y)$,

$$R(x) = \widetilde{R}(\pi(x)) = \widetilde{R}(\pi(y)) = R(y).$$

Thus R is constant on orbits. *(ii)⇒(i).* If R is constant on orbits, then for every $g \in G$ and $x \in X$, the points x and $g \cdot x$ lie in the same orbit, so

$$R(g \cdot x) = R(x).$$

Hence R is coherent. □

Remark 3.2.7 (Structural meaning of quotient semantics). The orbit projection

$$\pi : X \to \mathsf{Phys} = X/G$$

is not merely a convenient quotient map. By Theorem 3.2.5, it is universal among coherent descriptions of X. Thus quotient semantics is determined by diagonal redundancy: every admissible state report factors uniquely through Phys, and no finer representative-level distinction survives without additional structure.

Remark 3.2.8 (Categorical formulation). Let $\mathsf{Inv}(X)$ be the category whose objects are pairs (S, R), where S is a set and $R : X \to S$ is coherent, and whose morphisms

$$f : (S, R) \to (S', R')$$

are maps $f : S \to S'$ satisfying

$$R' = f \circ R.$$

Then (Phys, π) is an initial object of $\mathsf{Inv}(X)$. This is precisely the categorical form of Theorem 3.2.5.

3.3 Pure orbit loops

We now analyze histories that move entirely inside a single diagonal orbit.

Definition 3.3.1 (Pure diagonal orbit loop)**.** Fix $T > 0$. A *pure diagonal orbit loop* is a map

$$\Gamma : [0, T] \to X$$

for which there exists a map

$$g : [0, T] \to G$$

such that

$$g(0) = g(T) = e \qquad \text{and} \qquad \Gamma(t) = g(t) \cdot \Gamma(0) \quad \text{for all } t \in [0, T].$$

Definition 3.3.2 (A-supported and B-supported representatives)**.** Let $\Gamma : [0, T] \to X$ be a path. An *A-supported representative* of Γ is a path

$$\Gamma_A(t) = (x_A(t), x_B(0))$$

such that

$$\pi(\Gamma_A(t)) = \pi(\Gamma(t)) \qquad \text{for all } t \in [0, T].$$

Similarly, a *B-supported representative* is a path

$$\Gamma_B(t) = (x_A(0), x_B(t))$$

satisfying

$$\pi(\Gamma_B(t)) = \pi(\Gamma(t)) \qquad \text{for all } t \in [0, T].$$

Lemma 3.3.3 (Single-orbit property)**.** *If Γ is a pure diagonal orbit loop, then*

$$\pi(\Gamma(t)) = \pi(\Gamma(0)) \qquad \textit{for all } t \in [0, T].$$

Proof. For each $t \in [0, T]$, the defining relation

$$\Gamma(t) = g(t) \cdot \Gamma(0)$$

shows that $\Gamma(t)$ lies in the same G-orbit as $\Gamma(0)$. Since π is the orbit projection, it takes equal values on orbit-equivalent points. Therefore

$$\pi(\Gamma(t)) = \pi(\Gamma(0))$$

for all t. □

Remark 3.3.4. A pure orbit loop may be descriptively nontrivial in the product presentation $X_A \times X_B$, but orbit-theoretically it is trivial: it remains inside a single orbit of the diagonal action.

3.4 Orbit-equivalent supported representatives

The following proposition isolates the elementary symmetry relating supported representatives of the same quotient path.

Proposition 3.4.1 (Orbit-equivalent supported representatives)**.** *Let $\Gamma : [0,T] \to X$ be a pure diagonal orbit loop with witness $g : [0,T] \to G$, and assume Γ admits an A-supported representative Γ_A. Define*

$$\widetilde{\Gamma}(t) := g(t)^{-1} \cdot \Gamma_A(t).$$

Then:

(i) *$\widetilde{\Gamma}$ is a path in X satisfying*

$$\pi(\widetilde{\Gamma}(t)) = \pi(\Gamma(t)) \qquad \text{for all } t \in [0,T];$$

(ii) *one has the reconstruction identity*

$$\Gamma_A(t) = g(t) \cdot \widetilde{\Gamma}(t) \qquad \text{for all } t \in [0,T];$$

(iii) *the property of admitting an A-supported representative depends only on the quotient path $t \mapsto \pi(\Gamma(t))$;*

(iv) *if a B-supported representative of the same quotient path is chosen, then both Γ and $\widetilde{\Gamma}$ are pointwise orbit-equivalent to it.*

Proof. For each $t \in [0,T]$,

$$\widetilde{\Gamma}(t) = g(t)^{-1} \cdot \Gamma_A(t)$$

lies in the same G-orbit as $\Gamma_A(t)$. Since Γ_A is a representative of Γ, we have

$$\pi(\Gamma_A(t)) = \pi(\Gamma(t)).$$

Therefore

$$\pi(\widetilde{\Gamma}(t)) = \pi(\Gamma_A(t)) = \pi(\Gamma(t)),$$

which proves (i). Statement (ii) is immediate from the definition:

$$g(t) \cdot \widetilde{\Gamma}(t) = g(t) \cdot \big(g(t)^{-1} \cdot \Gamma_A(t)\big) = \Gamma_A(t).$$

For (iii), the existence of an A-supported representative is, by definition, a statement about whether the quotient path $t \mapsto \pi(\Gamma(t))$ can be realized by a path of the form $(x_A(t), x_B(0))$. Thus it depends only on the quotient path itself. For (iv), let Γ_B be a B-supported representative of the same quotient path. Then

$$\pi(\Gamma_B(t)) = \pi(\Gamma(t)) = \pi(\widetilde{\Gamma}(t)) \qquad \text{for all } t,$$

so $\Gamma_B(t)$, $\Gamma(t)$, and $\widetilde{\Gamma}(t)$ are pairwise orbit-equivalent at each time. □

3.5 Subsystem attribution obstruction

We now prove the main no-go statement.

Theorem 3.5.1 (No-go theorem)**.** *Let $R : X \to S$ be coherent, and let $\Gamma : [0, T] \to X$ be a pure diagonal orbit loop. Then*

$$R \circ \Gamma$$

is constant on $[0, T]$. Moreover, if Γ admits an A-supported representative and $\widetilde{\Gamma}$ is the associated path from Theorem 3.4.1, then

$$R(\widetilde{\Gamma}(t)) = R(\Gamma(t)) \qquad \textit{for all } t \in [0, T].$$

Proof. By Theorem 3.2.4, there exists a unique map

$$\widetilde{R} : \mathsf{Phys} \to S$$

such that

$$R = \widetilde{R} \circ \pi.$$

Since Γ is a pure diagonal orbit loop, Theorem 3.3.3 gives

$$\pi(\Gamma(t)) = \pi(\Gamma(0)) \qquad \text{for all } t \in [0, T].$$

Hence for all t,

$$R(\Gamma(t)) = \widetilde{R}(\pi(\Gamma(t))) = \widetilde{R}(\pi(\Gamma(0))),$$

which is independent of t. Thus $R \circ \Gamma$ is constant. For the second statement, Theorem 3.4.1(i) gives

$$\pi(\widetilde{\Gamma}(t)) = \pi(\Gamma(t)) \qquad \text{for all } t \in [0, T].$$

Therefore

$$R(\widetilde{\Gamma}(t)) = \widetilde{R}(\pi(\widetilde{\Gamma}(t))) = \widetilde{R}(\pi(\Gamma(t))) = R(\Gamma(t)).$$

□

Corollary 3.5.2 (No subsystem attribution at quotient level)**.** *Within quotient semantics, subsystem attribution for pure orbit motion does not define an invariant: every coherent report is constant along pure orbit loops, and any two pointwise orbit-equivalent representatives have identical coherent report values.*

Proof. The constancy statement is the first part of Theorem 3.5.1. The pointwise orbit-equivalence statement follows from the second part and Theorem 3.4.1(iv). □

3.6 Orbit-preserving maps

Theorem 3.5.1 is the pathwise case of a more general orbit-invisibility principle.

Definition 3.6.1 (Orbit-preserving map)**.** A map

$$\Theta_X : X \to X$$

is *orbit-preserving* if

$$\pi \circ \Theta_X = \pi.$$

Equivalently, $\Theta_X(x)$ lies in the same G-orbit as x for every $x \in X$.

Theorem 3.6.2 (Orbit-preserving maps are coherently trivial)**.** *Let*

$$\Theta_X : X \to X$$

be orbit-preserving. Then for every coherent report $R : X \to S$,

$$R \circ \Theta_X = R.$$

Proof. By Theorem 3.2.4, write

$$R = \widetilde{R} \circ \pi$$

for a unique map $\widetilde{R} : \mathsf{Phys} \to S$. Then for every $x \in X$,

$$R(\Theta_X(x)) = \widetilde{R}(\pi(\Theta_X(x))) = \widetilde{R}(\pi(x)) = R(x),$$

because Θ_X is orbit-preserving. □

Remark 3.6.3. Within quotient semantics, an orbit-preserving map introduces no new invariant information. Any nontrivial distinction must therefore arise from structure not already encoded by the quotient projection.

3.7 Finite protocol extensions and the two-locus classification

We now classify all ways of enriching quotient semantics on finite comparison protocols.

Definition 3.7.1 (Finite protocol network)**.** A *finite protocol network* is a finite directed graph

$$\mathcal{N} = (V, E)$$

whose vertex set satisfies

$$V \subseteq \mathsf{Phys}.$$

Write $\mathrm{Path}(\mathcal{N})$ for the free category on $\mathcal{N}$.

Definition 3.7.2 (Extension of quotient semantics)**.** Let $\mathcal{N} = (V, E)$ be a finite protocol network. An *extension of quotient semantics* consists of the following data.

(i) *State compatibility:* for each $p \in V$, an object assignment

$$p \longmapsto F_{\mathcal{N}}(p)$$

depending only on $p \in \mathsf{Phys}$;

(ii) *Compositionality:* a small category

$$\mathsf{C}_{\mathcal{N}}$$

with object set V, together with a functor

$$F_{\mathcal{N}} : \mathrm{Path}(\mathcal{N}) \to \mathsf{C}_{\mathcal{N}}$$

extending the object assignment.

Definition 3.7.3 (Endpoint-determined extension)**.** An extension of quotient semantics is *endpoint-determined* if for every finite protocol network $\mathcal{N}$ and every pair of co-terminal paths

$$\gamma_1, \gamma_2 \in \mathrm{Path}(\mathcal{N})$$

one has

$$F_{\mathcal{N}}(\gamma_1) = F_{\mathcal{N}}(\gamma_2).$$

Definition 3.7.4 (Section)**.** A *section* of the orbit projection is a map

$$s : \mathsf{Phys} \to X$$

such that

$$\pi \circ s = \mathrm{id}_{\mathsf{Phys}} .$$

Theorem 3.7.5 (Two-locus classification within quotient-semantic extensions)**.** *Under the standing principle Standing Principle 1, let F be an* extension of quotient semantics *in the sense of Theorem 3.7.2. If F is not endpoint-determined, then at least one of the following two enrichment mechanisms occurs:*

(i) Representative selection: *a section*

$$s : \mathsf{Phys} \to X$$

of π is supplied;

(ii) Morphism enrichment: *there exist a network*

$$\mathcal{N} = (V, E)$$

and co-terminal paths

$$\gamma_1, \gamma_2 \in \mathrm{Path}(\mathcal{N})$$

such that

$$F_{\mathcal{N}}(\gamma_1) \neq F_{\mathcal{N}}(\gamma_2).$$

No third locus exists within the data specified in Theorem 3.7.2.

Proof. State compatibility fixes the object-level assignment by the points $p \in \mathsf{Phys}$ alone. Thus non-endpoint behavior cannot arise from a new interpretation of vertices as such. Accordingly, any failure of endpoint determinacy must be witnessed through at least one of two enrichment layers, and both may be present together. First, one may adjoin representative-level data lifting points of Phys back to X. This is exactly the supply of a section

$$s : \mathsf{Phys} \to X,$$

which chooses a distinguished representative in each orbit. Second, one may assign genuinely distinct morphism data to co-terminal paths. Since $\mathrm{Path}(\mathcal{N})$ is the free category on the graph $\mathcal{N}$, a functor out of it is determined by its values on objects, generating edges, and composition. The object layer is already fixed by state compatibility. Therefore any remaining failure of endpoint determinacy occurs at the morphism layer, whether or not a section is also present. This proves that mechanisms (i) and (ii) exhaust all possible sources of non-endpoint behavior. No third locus exists because Theorem 3.7.2 contains no further data. □

Remark 3.7.6 (Object–morphism complementarity). The two enrichment mechanisms act on categorically orthogonal layers. Sections modify the representative choice at the object level. Transport or holonomy data modify the morphism layer while leaving the object set unchanged. Under the axioms of Theorem 3.7.2, nothing else is available.

3.8 Transport schemes and holonomy

We now recast the morphism-locus part of the classification as route-dependent transport.

Definition 3.8.1 (Free path groupoid)**.** Let $\mathcal{N} = (V, E)$ be a finite directed graph. Write

$$\mathrm{Path}^{\pm}(\mathcal{N})$$

for the free groupoid obtained from $\mathcal{N}$ by adjoining formal inverses to all directed edges.

Definition 3.8.2 (Comparison groupoid)**.** A *comparison groupoid* on $\mathcal{N}$ is a small groupoid

$$\mathsf{C}_{\mathcal{N}} \rightrightarrows V.$$

Definition 3.8.3 (Transport scheme)**.** A *transport scheme* on a finite network $\mathcal{N} = (V, E)$ consists of a choice, for each edge $e : p \to q$, of a morphism

$$\tau_e \in \mathrm{Hom}_{\mathsf{C}_{\mathcal{N}}}(p, q).$$

By the universal property of the free groupoid, these data extend uniquely to a functor

$$\mathrm{Hol} : \mathrm{Path}^{\pm}(\mathcal{N}) \to \mathsf{C}_{\mathcal{N}}$$

satisfying

$$\mathrm{Hol}(e) = \tau_e \qquad \text{for every generating edge } e.$$

Definition 3.8.4 (Route dependence)**.** A transport scheme is *endpoint-determined* if for any two morphisms

$$\gamma_1, \gamma_2 : p \to q$$

in $\mathrm{Path}^{\pm}(\mathcal{N})$ one has

$$\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2).$$

Otherwise it is *route-dependent*.

Lemma 3.8.5 (Loop reduction)**.** *Let*

$$\gamma_1, \gamma_2 : p \to q$$

be morphisms in $\mathrm{Path}^{\pm}(\mathcal{N})$*. Then*

$$\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2) \iff \mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{id}_p\,.$$

Proof. Because $\mathsf{C}_{\mathcal{N}}$ is a groupoid, $\mathrm{Hol}(\gamma_2)$ is invertible. Therefore

$$\mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{Hol}(\gamma_2)^{-1} \circ \mathrm{Hol}(\gamma_1).$$

Thus $\mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{id}_p$ if and only if

$$\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2).$$

□

Theorem 3.8.6 (Route dependence $\iff$ loop defect)**.** *A transport scheme is route-dependent if and only if there exists a based loop*

$$\ell : p \to p$$

in $\mathrm{Path}^{\pm}(\mathcal{N})$ *such that*

$$\mathrm{Hol}(\ell) \neq \mathrm{id}_p\,.$$

Proof. Suppose first that the transport scheme is route-dependent. Then there exist co-terminal morphisms

$$\gamma_1, \gamma_2 : p \to q$$

such that

$$\mathrm{Hol}(\gamma_1) \neq \mathrm{Hol}(\gamma_2).$$

Set

$$\ell := \gamma_2^{-1} \circ \gamma_1.$$

Then by Theorem 3.8.5,

$$\mathrm{Hol}(\ell) = \mathrm{Hol}(\gamma_2)^{-1} \circ \mathrm{Hol}(\gamma_1) \neq \mathrm{id}_p\,.$$

Conversely, suppose every based loop has trivial holonomy. Let

$$\gamma_1, \gamma_2 : p \to q$$

be co-terminal. Then

$$\gamma_2^{-1} \circ \gamma_1$$

is a loop at p, so

$$\mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{id}_p .$$

By Theorem 3.8.5,

$$\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2).$$

Thus the scheme is endpoint-determined. □

Corollary 3.8.7 (Transport form of the two-locus classification)**.** *For finite protocol extensions in the sense of Theorem 3.7.2, any enrichment beyond quotient semantics must occur through at least one of the following classified loci, and may occur through both:*

(i) *object-level enrichment by a section of* π*;*

(ii) *morphism-level enrichment by a route-dependent transport scheme, equivalently by nontrivial loop defect.*

Proof. Combine Theorem 3.7.5 with Theorem 3.8.6. □

Remark 3.8.8 (Positive interpretation)*.* The two-locus classification says exactly where additional structure can enter once quotient semantics has been fixed. A section chooses preferred representatives in each orbit. A route-dependent transport scheme introduces morphism-level data not determined by the endpoints in Phys. Absent both of these two enrichments, no further invariant distinction exists beyond orbit-level content.

3.9 Conclusion

Starting from the canonical diagonal redundancy data produced in chapter 2,

$$X = X_A \times X_B, \qquad G = \mathrm{Aut}(U, \mathcal{C}), \qquad \pi : X \to \mathsf{Phys} := X/G,$$

the semantic consequences of diagonal symmetry are now fixed. First, admissible closed-system reports are exactly the coherent maps

$$R : X \to S$$

that are invariant under the diagonal action of G. By the descent theorem, every such report factors uniquely through the orbit projection

$$\pi : X \to \mathsf{Phys}.$$

Closed-system semantics is therefore determined to be quotient semantics: all invariant reportable content is carried by the orbit space Phys. Second, pure orbit motion is invisible to coherent reports. Every coherent report is constant along a pure diagonal orbit loop, and any pointwise orbit-equivalent supported representatives of such a loop have identical coherent report values. Hence subsystem attribution for pure orbit motion does not define a quotient-level invariant. Third, the possible enrichments of quotient semantics on finite comparison protocols are completely classified. Within the extension formalism of Theorem 3.7.2, the classification is exhaustive: if an extension is not endpoint-determined, then additional structure enters through one or both of the following mechanisms: representative selection through a section of π, and morphism-level transport data, equivalently nontrivial loop defect. There is no third locus within that formalism. The chapter therefore identifies the precise boundary of closed quotient semantics. Diagonal redundancy collapses all intrinsic invariant content of the closed comparison world to the orbit quotient

$$\mathsf{Phys} = X/G.$$

Any further distinction must therefore be imported explicitly through at least one of the two structural mechanisms above, and possibly through both together. Accordingly, the semantic frontier is exact: intrinsic closed-system content is exhausted by quotient data, and every admissible refinement is confined to the classified object and morphism loci.

Chapter 4 compresses this semantic arc into a single closure theorem that functions as the entry statement for the subsequent classification and obstruction chapters.

Chapter 4

Closed Systems from Comparison Completeness

4.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter consolidates the compactness/closure/quotient consequences of Chapters 1–3 into a single closure theorem that serves as the formal entry point to the representative-enrichment analysis in chapter 5. Chapter 1 isolated the Boolean compactness mechanism governing passage from finite-stage admissibility to global admissibility. Chapter 2 applied that mechanism to the primitive comparison world

$$(U, \mathcal{C}),$$

proved that rectangular comparison completeness is equivalent to bijectivity of the canonical factor map

$$\Theta : U \to X_A \times X_B, \qquad X_A := U/\alpha, \qquad X_B := U/\beta,$$

and thereby obtained the canonical product presentation

$$U \cong X_A \times X_B.$$

Chapter 3 then showed that under this determined product presentation the intrinsic symmetry group

$$G := \mathrm{Aut}(U, \mathcal{C})$$

acts diagonally on

$$X := X_A \times X_B,$$

that the comparison world canonically determines the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G,$$

and that admissible closed-system reports are exactly the coherent maps on X, equivalently the maps descending uniquely through π. The present chapter does not introduce new primitive structure. Its purpose is to identify the single internal condition responsible for the entire Part II spine and to fix, at theorem level, the meaning of the phrase *closed system* in the present framework. From this point onward the convention is exact:

$$\text{closed} \quad := \quad \text{rectangularly complete.}$$

This is a terminological identification, not a new axiom. The content of the chapter is that this one internal completeness condition is exactly what determines the canonical product presentation, diagonal redundancy, the orbit quotient, and quotient semantics. Accordingly, the chapter has three tasks:

(i) to define closedness exactly as the internal profile-pairing completeness condition isolated in Chapter 2;

(ii) to prove that closedness is exactly the condition determining the canonical product presentation and the quotient-semantic structure of Chapter 3;

(iii) to prove that closedness is minimal for determining this structure uniformly across comparison worlds.

The first two tasks are consequences of Chapters 2 and 3. The genuinely new content of the present chapter is the minimality theorem and the compression of Part II into a single classification statement. Thus this is the point at which the phrase *closed system* ceases to be heuristic. Within the present framework, a closed system is exactly a comparison world whose intrinsic left-profile and right-profile data are internally complete.

4.2 Closure as profile-pairing completeness

Recall from Theorems 2.3.2 and 2.4.1 that every comparison world $(U, \mathcal{C})$ determines intrinsic congruences α, β, quotient sets

$$X_A := U/\alpha, \qquad X_B := U/\beta,$$

and the canonical factor map

$$\Theta : U \to X_A \times X_B, \qquad \Theta(u) = ([u]_\alpha, [u]_\beta).$$

Definition 4.2.1 (Closure)**.** A comparison world $(U, \mathcal{C})$ is *closed* if it is rectangularly complete in the sense of Theorem 2.5.1; equivalently, if for every pair

$$A \in X_A, \qquad B \in X_B,$$

there exists a unique element $u \in U$ such that

$$[u]_\alpha = A, \qquad [u]_\beta = B.$$

Remark 4.2.2 (Terminological status). Theorem 4.2.1 introduces no new mathematical structure. It merely renames the rectangular comparison completeness condition of Theorem 2.5.1 in the language appropriate to its semantic role in the remainder of the manuscript.

Remark 4.2.3 (Internality of closure). Closure is purely internal. It refers neither to a geometric boundary nor to an ambient environment, an external observer, or a background space. It asserts only that every admissible pairing of left and right comparison profiles is realized by a unique state of the world itself.

Proposition 4.2.4 (Closure $\iff$ bijectivity of the canonical profile map). *For a comparison world* $(U, \mathcal{C})$*, the following are equivalent:*

(i) $(U, \mathcal{C})$ *is closed.*

(ii) *The canonical factor map*

$$\Theta : U \to X_A \times X_B$$

is bijective.

When these conditions hold, the identification

$$U \cong X_A \times X_B$$

is canonical, namely given by Θ*.*

Proof. By Theorem 4.2.1, closedness is exactly rectangular completeness. By Theorem 2.6.1, rectangular completeness is equivalent to bijectivity of Θ, and the same theorem records that the resulting identification of U with $X_A \times X_B$ is canonical. □

Remark 4.2.5 (Closure as absence of profile-pair defects). Failure of closure means that at least one of two profile-pair defects occurs: either Θ is not injective, so distinct states determine the same left-right profile pair, or Θ is not surjective, so some left-right profile pair is not realized by any state; both defects may occur simultaneously. Thus closure is precisely the absence of profile-pair defects.

4.3 Closure determines diagonal redundancy and quotient semantics

Once closure holds, the product structure and quotient-semantic architecture of Part II are determined. Assume henceforth, under item (SP3), that the equivalent conditions of Theorem 4.2.4 hold. Write

$$X := X_A \times X_B$$

and identify U with X through the canonical bijection Θ.

Definition 4.3.1 (Determined diagonal redundancy)**.** A comparison world $(U, \mathcal{C})$ *determines diagonal redundancy* if the following data are canonically determined from $(U, \mathcal{C})$ without any additional primitive:

(i) the factor spaces
$$X_A = U/\alpha, \qquad X_B = U/\beta;$$

(ii) the canonical identification
$$U \cong X_A \times X_B;$$

(iii) the canonical diagonal action of
$$G := \mathrm{Aut}(U, \mathcal{C})$$
on
$$X = X_A \times X_B.$$

Proposition 4.3.2 (Closure determines diagonal redundancy)**.** *If $(U, \mathcal{C})$ is closed, then it determines diagonal redundancy in the sense of Theorem 4.3.1. More precisely, under the canonical identification $U \cong X_A \times X_B$, the natural action of*
$$G = \mathrm{Aut}(U, \mathcal{C})$$
on U becomes the diagonal action
$$g \cdot (x_A, x_B) := (g \cdot x_A,\ g \cdot x_B)$$
on $X_A \times X_B$.

Proof. By Theorem 4.2.4, closure yields the canonical identification
$$U \cong X_A \times X_B$$
through Θ. The G-invariance of α and β, proved in Theorem 2.3.5, gives well-defined induced actions on X_A and X_B; see Theorems 2.8.1 and 2.8.2. By Theorem 2.8.3, the action of G on U is transported by Θ exactly to the diagonal action on $X_A \times X_B$. □

Corollary 4.3.3 (Canonical quotient input)**.** *If $(U, \mathcal{C})$ is closed, then the orbit quotient*
$$\mathsf{Phys} := X/G$$
and the orbit projection
$$\pi : X \to \mathsf{Phys}$$
are canonically determined by the comparison world.

Proof. By Theorem 4.3.2, closure determines the diagonal action of G on X. The orbit quotient and its projection are therefore canonically determined. □

Proposition 4.3.4 (Universal quotient description of coherent reports)**.** *Under the standing principle Standing Principle 1, let $(U, \mathcal{C})$ be closed, and let*

$$\pi : X \to \mathsf{Phys} := X/G$$

be the orbit projection for the determined diagonal action. Then for every set S, composition with π induces a bijection

$$\pi^* : \mathrm{Map}(\mathsf{Phys}, S) \xrightarrow{\cong} \mathrm{Map}_G(X, S),$$

where

$$\mathrm{Map}_G(X, S) := \{R : X \to S : \ R(g \cdot x) = R(x) \ \forall g \in G, \ x \in X\}$$

is the set of coherent, equivalently G-invariant, maps.

Proof. Define

$$\pi^*(f) := f \circ \pi \qquad \text{for } f \in \mathrm{Map}(\mathsf{Phys}, S).$$

Because $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$, the composite $f \circ \pi$ is G-invariant. Hence π^* is well defined. Let

$$R \in \mathrm{Map}_G(X, S).$$

Then R is coherent in the sense of Theorem 3.2.1. By Theorem 3.2.4, there exists a unique map

$$\widetilde{R} : \mathsf{Phys} \to S$$

such that

$$R = \widetilde{R} \circ \pi.$$

Therefore

$$R = \pi^*(\widetilde{R}),$$

so π^* is surjective. Now suppose

$$\pi^*(f_1) = \pi^*(f_2), \qquad \text{i.e.} \qquad f_1 \circ \pi = f_2 \circ \pi.$$

Let $p \in \mathsf{Phys}$. Since π is surjective, choose $x \in X$ with

$$\pi(x) = p.$$

Then

$$f_1(p) = f_1(\pi(x)) = (f_1 \circ \pi)(x) = (f_2 \circ \pi)(x) = f_2(\pi(x)) = f_2(p).$$

Hence $f_1 = f_2$, so π^* is injective. Therefore π^* is a bijection. □

Corollary 4.3.5 (Closed-system semantics is quotient semantics)**.** *Under closure, admissible state reports in the closed-system sense are exactly maps on the quotient state space*

$$\mathsf{Phys} = X/G.$$

Proof. By Theorem 3.2.2, admissible state reports are exactly coherent maps, equivalently the elements of $\mathrm{Map}_G(X, S)$ for varying codomain S. By Theorem 4.3.4, these are in canonical bijection with maps $\mathsf{Phys} \to S$. □

Remark 4.3.6 (Why quotient semantics is not an extra axiom). The quotient Phys is not introduced by a separate gauge principle. Once closure determines the canonical product structure and diagonal redundancy, the universal property of π determines the semantic identification of admissible reports with maps on Phys.

4.4 Logical minimality of closure

The genuinely new content of the chapter is the minimality statement: closure is not merely sufficient for the quotient-semantic spine. It is the minimal internal condition that determines it uniformly.

Theorem 4.4.1 (Logical minimality of closure)**.** *Let P be any property of comparison worlds with the following uniform determining property: for every comparison world $(U, \mathcal{C})$ satisfying P, the world canonically admits an identification*

$$U \cong X_A \times X_B$$

compatible with the intrinsic congruences α and β. Then P implies rectangular completeness. Equivalently, P implies closedness.

Proof. Fix a comparison world $(U, \mathcal{C})$ satisfying P. By hypothesis, $(U, \mathcal{C})$ canonically admits an identification

$$U \cong X_A \times X_B$$

compatible with the intrinsic congruences α and β. By compatibility, the canonical factor map

$$\Theta : U \to X_A \times X_B$$

must coincide with that identification, hence is bijective. By Theorem 2.6.1, bijectivity of Θ is equivalent to rectangular completeness. Thus P implies closedness. □

Theorem 4.4.2 (Minimality of closure for the quotient-semantic spine)**.** *For a comparison world $(U, \mathcal{C})$, the following are equivalent:*

(1) *$(U, \mathcal{C})$ is closed.*

(2) *The canonical map*

$$\Theta : U \to X_A \times X_B$$

is bijective.

(3) *$(U, \mathcal{C})$ determines diagonal redundancy.*

Moreover, no strictly weaker closure property guarantees determined diagonal redundancy for all comparison worlds.

Proof. The equivalence (1)$\Longleftrightarrow$(2) is Theorem 4.2.4. The implication (2)$\Longrightarrow$(3) is Theorem 4.3.2. Conversely, if diagonal redundancy is determined, then by Theorem 4.3.1 the comparison world canonically determines an identification

$$U \cong X_A \times X_B.$$

Since Θ is the canonical map from U to that same product, it must be bijective. Hence (3)$\Longrightarrow$(2). If P is any property of comparison worlds which uniformly determines diagonal redundancy, then P satisfies the hypothesis of Theorem 4.4.1. Therefore P implies closedness. Hence no strictly weaker property can force diagonal redundancy uniformly. □

Corollary 4.4.3 (Logical irreversibility of the spine)**.** *Once closure holds, the structural chain*

$$\begin{aligned}(U, \mathcal{C}) &\Longrightarrow U \cong X_A \times X_B \\ &\Longrightarrow G \curvearrowright X \ \textit{diagonally} \\ &\Longrightarrow \pi : X \to \mathsf{Phys} = X/G \\ &\Longrightarrow \textit{canonical descent} \\ &\Longrightarrow \textit{subsystem attribution obstruction}\end{aligned}$$

is determined. No weaker closure axiom guarantees this chain uniformly.

Proof. The first three steps follow from Theorems 4.2.4, 4.3.2 and 4.3.3. Canonical descent is the universal quotient description of Theorem 4.3.4. The final obstruction statement is Theorem 3.5.1. The minimality clause is the second assertion of Theorem 4.4.2. □

4.5 Part II classification theorem

The results of Chapters 2, 3, and the present chapter now compress into a single classification theorem.

Theorem 4.5.1 (Closed-system classification from comparison completeness)**.** *Under the standing principle Standing Principle 1, let $(U, \mathcal{C})$ be a comparison world in the local distinguishability regime*

$$\alpha \cap \beta = \Delta_U$$

and satisfying the closure clause item (SP3)*, equivalently rectangular comparison completeness. Then the following structure is canonically determined.*

(i) *the canonical factor map*

$$\Theta : U \to X_A \times X_B, \qquad X_A := U/\alpha, \qquad X_B := U/\beta,$$

is bijective, hence

$$U \cong X_A \times X_B;$$

(ii) *the intrinsic symmetry group*

$$G := \mathrm{Aut}(U, \mathcal{C})$$

acts diagonally on

$$X := X_A \times X_B,$$

so the comparison world canonically determines the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G;$$

(iii) *for every set S, pullback along π induces a bijection*

$$\mathrm{Map}(\mathsf{Phys}, S) \xrightarrow{\cong} \mathrm{Map}_G(X, S),$$

where $\mathrm{Map}_G(X, S)$ denotes the coherent, equivalently G-invariant, maps $X \to S$;

(iv) *consequently, admissible closed-system content is exactly quotient content on* Phys*;*

(v) *for pure orbit motion, coherent reports are constant, so subsystem attribution along pure diagonal orbit loops is not a quotient-level invariant;*

(vi) *any extension of quotient semantics on finite comparison protocols that is not endpoint-determined introduces additional structure through at least one of two loci, possibly both, and through no others:*

(a) *object-level representative choice, i.e. a section*

$$s : \mathsf{Phys} \to X;$$

(b) *morphism-level transport data, equivalently nontrivial loop defect.*

Hence a closed system in the present sense is canonically classified by the quotient-semantic datum

$$X_A,\ X_B,\ G,\ \pi : X_A \times X_B \to (X_A \times X_B)/G,$$

together with the theorem that all admissible content descends to the quotient and all non-quotient enrichment enters through at least one of the two classified loci, possibly both, with no third locus.

Proof. Item (i) is Theorem 4.2.4. Item (ii) is Theorem 4.3.2 together with Theorem 4.3.3. Item (iii) is Theorem 4.3.4. Item (iv) is Theorem 4.3.5. Item (v) is Theorem 3.5.1. Item (vi) is Theorems 3.7.5 and 3.8.7. □

4.6 Interpretation

The formal results admit an exact internal reading.

Remark 4.6.1 (Closure as internal completeness)*.* Within the present framework, the term *closed system* has a purely comparison-theoretic meaning. It does not refer to a geometric boundary or to an externally imposed isolation condition. Rather, closure means that the comparison structure already realizes every state required by its own left-profile and right-profile data. Equivalently, closure is the condition that every pair consisting of an α-class and a β-class is realized by a unique element of U. This is exactly the rectangular comparison completeness condition of Theorem 2.5.1, recast terminologically in Theorem 4.2.1. Thus closedness is not added from outside. It is the statement that the comparison world is internally complete with respect to its own profile data. Once this internal completeness holds, the canonical product decomposition, the diagonal symmetry, the orbit quotient, and the quotient semantics all follow as necessary consequences of the comparison structure itself.

Remark 4.6.2 (Structural centrality of the chapter)*.* Everything later in the stack presupposes the theorem-level content recorded here. The transport theory of enrichment, the stabilization of loop defect, and the later curvature realization of that same obstruction all require that admissible state content has already been determined down to quotient semantics. That determining is exactly the content of closure as established in the present chapter.

4.7 Conclusion

The structural content of Part II is now complete. A comparison world is closed exactly when it satisfies rectangular comparison completeness. Equivalently, it is closed exactly when the canonical factor map

$$\Theta : U \to X_A \times X_B$$

is bijective. In that case the comparison structure determines the canonical product presentation

$$U \cong X_A \times X_B,$$

the diagonal action of

$$G = \mathrm{Aut}(U, \mathcal{C})$$

on

$$X = X_A \times X_B,$$

and hence the canonical orbit projection

$$\pi : X \to \mathsf{Phys} = X/G.$$

The universal property of π then identifies admissible closed-system reports with maps on Phys. Quotient semantics is therefore not an added axiom. It is a theorem-level consequence of closure. Moreover, closure is minimal for this purpose. No strictly weaker condition determines diagonal redundancy uniformly, and hence no strictly weaker condition determines the quotient-semantic spine of Part II. This is the structural role of the chapter. It closes Part II by fixing the exact theorem-level meaning of a closed system and by proving that the passage from primitive comparison data to canonical quotient semantics is determined exactly by closure, i.e. by rectangular comparison completeness. Accordingly, Part II concludes at theorem level: closure is both necessary and sufficient for the quotient-semantic backbone, and no weaker hypothesis supports the same uniform determination.

From this fixed backbone, chapter 5 undertakes the universal classification of admissible enrichment mechanisms beyond quotient semantics.

Part III

Lifting, Transport, and Holonomy

Chapter 5

Universal Classification of Representative Enrichment

5.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the representative-enrichment consequences of the quotient-descend and transport-visibility clauses (items (SP4) and (SP5)). With the structural closure theorem of chapter 4 fixed, it develops the universal enrichment classification and proves that no third enrichment locus is admissible. Chapters 2 and 3 complete the semantic part of the closed comparison program. Rectangular comparison completeness yields the canonical factorization

$$U \cong X_A \times X_B =: X,$$

the intrinsic symmetry group

$$G := \mathrm{Aut}(U, \mathcal{C})$$

acts diagonally on X, and the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G$$

is thereby fixed by the comparison datum itself. By canonical descent, admissible state reports are exactly the G-invariant maps

$$R : X \to S,$$

equivalently the maps factoring uniquely through π. The quotient Phys therefore contains precisely the coherent state content visible to admissible observables. From this point onward the problem is no longer semantic. The quotient already records everything that survives intrinsic descent. Any passage back from Phys to X must therefore introduce data absent from quotient semantics itself. The possible enrichment data are exhausted by two loci that may occur together: the object layer, consisting of the choice of representatives in the fibres of

$$\pi : X \to \mathsf{Phys},$$

and the morphism layer, consisting of the transport elements of G relating those representatives along arrows. No third locus exists. This is not an interpretation imposed on the formalism. It is determined by the categorical object governing representative reconstruction. Once X, the diagonal G-action, and the quotient map π are fixed, representative reconstructions are precisely functors into the action groupoid

$$X /\!\!/ G,$$

and a functor into $X /\!\!/ G$ is determined exactly by its values on objects and on morphisms. The object and morphism loci are therefore the two irreducible layers of data carried by the lifting object itself. The chapter is organized around the following classification theorem.

Theorem 5.1.1 (Universal classification of representative enrichment)**.** *Under the standing principle Standing Principle 1, let*

$$\pi : X \to \mathsf{Phys} := X/G$$

be the canonical orbit projection of the diagonal action of

$$G = \mathrm{Aut}(U, \mathcal{C})$$

on the canonical carrier $X = X_A \times X_B$, and let

$$\mathrm{pr} : X /\!\!/ G \to \mathsf{Phys}^{\mathrm{disc}}$$

be the associated component functor. Let $\mathcal{I}$ be a small category, and let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient-level protocol. Then every representative reconstruction of $\overline{R}$ is exactly a functor

$$T : \mathcal{I} \to X /\!\!/ G \qquad \textit{with} \qquad \mathrm{pr} \circ T = \overline{R}.$$

Whenever such a lift exists, it is exhausted by the following two loci of data, which may occur together and admit no third independent enrichment class:

(i) Object locus: *for each object $i \in \mathrm{Ob}(\mathcal{I})$, a choice of representative*

$$x_i \in X \qquad \textit{with} \qquad \pi(x_i) = \overline{R}(i);$$

(ii) Morphism locus: *for each morphism $f : i \to j$ in $\mathcal{I}$, a transport element*

$$g_f \in G \qquad \textit{such that} \qquad g_f \cdot x_i = x_j,$$

subject to the functorial relations

$$g_{\mathrm{id}_i} = e, \qquad g_{f_2 \circ f_1} = g_{f_2} g_{f_1}.$$

These two loci determine the lift uniquely, and no third independent enrichment datum exists. Equivalently, the action groupoid $X /\!\!/ G$ is the universal and exhaustive recipient of all representative enrichments of quotient semantics.

The theorem isolates the only possible sources of non-quotient data. Representative choice occupies the object locus. Transport, holonomy, the first finite triangle obstruction, and the later loop-defect globalization occupy the morphism locus. The remainder of the chapter makes this statement precise. First the action groupoid is constructed and projected to the discrete quotient $\mathsf{Phys}^{\mathrm{disc}}$. Quotient protocols are then formulated as functors into $\mathsf{Phys}^{\mathrm{disc}}$, and representative reconstruction is expressed as a lifting problem. Existence is treated separately by a connected-component criterion. The classification theorem is then proved by direct analysis of the data carried by a functor into $X /\!\!/ G$. Only after the classification is in hand is the action groupoid reformulated by its universal property. Finally, the comparison-theoretic origin of the construction is identified: rectangular completeness is exactly the primitive that canonically produces the carrier X, the diagonal action of G, and hence the lifting groupoid itself. This is the point at which quotient semantics passes into transport algebra.

5.2 The action groupoid

Definition 5.2.1 (Action groupoid)**.** Let a group G act on a set X. The *action groupoid*

$$X /\!\!/ G$$

is the category defined as follows.

(1) The objects are the elements $x \in X$.

(2) For $x, y \in X$, a morphism $x \to y$ is a pair

$$(g, x)$$

with $g \in G$ such that $y = g \cdot x$.

(3) If

$$(g, x) : x \to g \cdot x \qquad \text{and} \qquad (h, g \cdot x) : g \cdot x \to hg \cdot x,$$

then their composite is

$$(h, g \cdot x) \circ (g, x) := (hg, x).$$

(4) The identity morphism at x is

$$\mathrm{id}_x := (e, x),$$

where $e \in G$ is the identity element.

Lemma 5.2.2. *The category $X /\!\!/ G$ is a groupoid.*

Proof. Associativity follows from associativity in G. For composable triples,

$$(k, hg\cdot x) \circ \big((h, g\cdot x) \circ (g, x)\big) = (k, hg\cdot x) \circ (hg, x) = (khg, x),$$

while

$$\big((k, hg\cdot x) \circ (h, g\cdot x)\big) \circ (g, x) = (kh, g\cdot x) \circ (g, x) = (khg, x).$$

The identity law is immediate:

$$(e, g\cdot x) \circ (g, x) = (eg, x) = (g, x), \qquad (g, x) \circ (e, x) = (ge, x) = (g, x).$$

Finally, every morphism $(g, x) : x \to g\cdot x$ is invertible, with inverse

$$(g^{-1}, g\cdot x) : g\cdot x \to x,$$

since

$$(g^{-1}, g\cdot x) \circ (g, x) = (e, x) = \mathrm{id}_x$$

and

$$(g, x) \circ (g^{-1}, g\cdot x) = (e, g\cdot x) = \mathrm{id}_{g\cdot x}\,.$$

Thus every morphism is invertible, so $X /\!\!/ G$ is a groupoid. □

Lemma 5.2.3 (Orbit characterization)**.** *For $x, y \in X$, the following are equivalent.*

(i) *x and y are isomorphic in $X /\!\!/ G$;*

(ii) *there exists $g \in G$ such that*

$$y = g\cdot x;$$

(iii)

$$\pi(x) = \pi(y),$$

where $\pi : X \to \mathsf{Phys} := X/G$ is the orbit projection.

Proof. A morphism $x \to y$ in $X /\!\!/ G$ is by definition a pair

$$(g, x) : x \to g\cdot x.$$

Hence such a morphism exists if and only if $y = g\cdot x$ for some $g \in G$. Since every morphism in $X /\!\!/ G$ is invertible by Theorem 5.2.2, this is equivalent to x and y being isomorphic. Thus (i) and (ii) are equivalent. Condition (ii) says exactly that x and y lie in the same G-orbit. By definition of the orbit projection, this is equivalent to $\pi(x) = \pi(y)$. Thus (ii) and (iii) are equivalent. □

5.3 The orbit quotient as a discrete base

Definition 5.3.1 (Discrete quotient category)**.** Let

$$\mathsf{Phys}^{\mathrm{disc}}$$

denote the discrete category whose objects are the elements of Phys and whose only morphisms are identity morphisms.

Definition 5.3.2 (Component functor)**.** Define

$$\mathrm{pr} : X /\!\!/ G \to \mathsf{Phys}^{\mathrm{disc}}$$

on objects by

$$\mathrm{pr}(x) := \pi(x),$$

and on morphisms by

$$\mathrm{pr}(g, x) := \mathrm{id}_{\pi(x)}\,.$$

Lemma 5.3.3. *The assignment* pr *is a functor.*

Proof. For identities,

$$\mathrm{pr}(\mathrm{id}_x) = \mathrm{pr}(e, x) = \mathrm{id}_{\pi(x)} = \mathrm{id}_{\mathrm{pr}(x)}\,.$$

For composition, let

$$(g, x) : x \to g \cdot x, \qquad (h, g \cdot x) : g \cdot x \to hg \cdot x.$$

Then

$$\mathrm{pr}\big((h, g \cdot x) \circ (g, x)\big) = \mathrm{pr}(hg, x) = \mathrm{id}_{\pi(x)}\,.$$

On the other hand,

$$\mathrm{pr}(h, g \cdot x) \circ \mathrm{pr}(g, x) = \mathrm{id}_{\pi(g \cdot x)} \circ \mathrm{id}_{\pi(x)}\,.$$

Since $\pi(g \cdot x) = \pi(x)$, the right-hand side is

$$\mathrm{id}_{\pi(x)}\,.$$

Thus pr preserves identities and composition. □

Remark 5.3.4. The functor pr forgets precisely the representative-level data that quotient semantics excludes: the chosen objects in X and the transport labels in G. What remains is the orbit class alone.

5.4 Quotient protocols and representative lifts

Definition 5.4.1 (Quotient-level protocol)**.** Let $\mathcal{I}$ be a small category. A *quotient-level protocol* is a functor

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}.$$

Because $\mathsf{Phys}^{\mathrm{disc}}$ is discrete, such a functor records only orbit classes attached to the objects of $\mathcal{I}$. It carries no representative choices and no transport information.

Definition 5.4.2 (Representative lift)**.** A *representative lift* of $\overline{R}$ is a functor

$$T : \mathcal{I} \to X /\!\!/ G$$

such that

$$\mathrm{pr} \circ T = \overline{R}.$$

Thus reconstruction of representatives from quotient-level data is a lifting problem

$$\begin{array}{ccc} & X /\!\!/ G & \\ T \nearrow & & \searrow \mathrm{pr} \\ \mathcal{I} & \xrightarrow{\overline{R}} & \mathsf{Phys}^{\mathrm{disc}}. \end{array}$$

Remark 5.4.3. This is the groupoid-level analogue of canonical descent. By Theorems 3.2.4 and 3.2.6, admissible maps out of X were exactly those descending through π. Here quotient-level data are lifted back through the canonical groupoid lying over $\mathsf{Phys}^{\mathrm{disc}}$.

5.5 Lift existence

Lemma 5.5.1 (Lift existence criterion)**.** *A representative lift of*

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

exists if and only if $\overline{R}$ is constant on each connected component of $\mathcal{I}$.

Proof. Assume first that a lift

$$T : \mathcal{I} \to X /\!\!/ G$$

exists. Let $i, j \in \mathrm{Ob}(\mathcal{I})$ lie in the same connected component of $\mathcal{I}$. Then there is a zig-zag of morphisms joining i to j. Applying T yields a zig-zag of morphisms in the groupoid $X /\!\!/ G$, so all objects appearing in the image zig-zag are isomorphic. By Theorem 5.2.3, their images under π coincide. Hence

$$\overline{R}(i) = \mathrm{pr}(T(i)) = \mathrm{pr}(T(j)) = \overline{R}(j).$$

Therefore $\overline{R}$ is constant on each connected component. Conversely, assume that $\overline{R}$ is constant on each connected component of $\mathcal{I}$. For each connected component C, choose an element

$$x_C \in X$$

such that

$$\pi(x_C) = \overline{R}(i) \qquad \text{for every } i \in C.$$

This is possible because π is surjective and $\overline{R}$ is constant on C. Define T on objects by

$$T(i) := x_C \qquad (i \in C).$$

For every morphism $f : i \to j$ inside the component C, define

$$T(f) := \mathrm{id}_{x_C}.$$

Then T preserves identities and composition because identities do. Moreover,

$$\mathrm{pr}(T(i)) = \pi(x_C) = \overline{R}(i),$$

so

$$\mathrm{pr} \circ T = \overline{R}.$$

Thus T is a representative lift. □

Remark 5.5.2. Because the target $\mathsf{Phys}^{\mathrm{disc}}$ is discrete, a connected quotient protocol can carry only one orbit per connected component. The criterion states that this obvious necessary condition is also sufficient.

5.6 Exhaustion of representative enrichment

Proof of Theorem 5.1.1. Let

$$T : \mathcal{I} \to X /\!\!/ G \qquad \text{with} \qquad \mathrm{pr} \circ T = \overline{R}$$

be a representative reconstruction. A functor into $X /\!\!/ G$ is determined uniquely by its values on objects and on morphisms. For each object $i \in \mathrm{Ob}(\mathcal{I})$, the value $T(i)$ is an element of X. Since

$$\mathrm{pr}(T(i)) = \overline{R}(i),$$

the point $T(i)$ is a representative of the orbit class $\overline{R}(i)$. This is exactly the object locus. For each morphism $f : i \to j$, the value $T(f)$ is a morphism in $X /\!\!/ G$ from $T(i)$ to $T(j)$. By definition of the action groupoid, such a morphism is of the form

$$T(f) = (g_f, T(i))$$

for some element $g_f \in G$ satisfying

$$g_f \cdot T(i) = T(j).$$

This chosen transport label is exactly the morphism locus. Without the later free-action hypothesis, it need not be determined by the endpoints alone. Functoriality imposes the relations

$$g_{\mathrm{id}_i} = e \qquad \text{and} \qquad g_{f_2 \circ f_1} = g_{f_2} g_{f_1}$$

for composable morphisms f_1, f_2. These are compatibility relations on the same transport labels; they do not introduce any further independent datum. Thus every representative reconstruction is determined completely by its object representatives and transport labels, and there is no third place at which additional representative-level structure can enter. This is exactly the statement of Theorem 5.1.1. □

Corollary 5.6.1 (Exact exhaustion)**.** *Every representative lift is exhausted by the following two loci of data, which may occur together:*

(i) *representative choice on objects;*

(ii) *transport assignment on morphisms, subject to functorial compatibility.*

No third independent invariant datum appears at the level of the action groupoid.

Proof. Immediate from Theorem 5.1.1. □

Theorem 5.6.2 (Free-action specialization: representative lifts as pointed flat G-torsor transport on the protocol)**.** *In the local free-fibre setup, assume that the orbit fibres of*

$$\pi : X \to \mathsf{Phys}$$

are free G-orbits. Let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient-level protocol, and let

$$T : \mathcal{I} \to X /\!\!/ G$$

be a representative lift. Then T determines, and is determined by, the following data:

(i) *for each object $i \in \mathrm{Ob}(\mathcal{I})$, the orbit fibre*

$$\pi^{-1}(\overline{R}(i)),$$

which is a G-torsor, together with a chosen representative point

$$x_i \in \pi^{-1}(\overline{R}(i));$$

(ii) *for each morphism $f : i \to j$, the torsor isomorphism*

$$\tau_f : \pi^{-1}(\overline{R}(i)) \to \pi^{-1}(\overline{R}(j))$$

induced by the transport label of $T(f)$, so that

$$\tau_f(x_i) = x_j;$$

(iii) *these isomorphisms satisfy the flatness relations*

$$\tau_{\mathrm{id}_i} = \mathrm{id}, \qquad \tau_{f_2 \circ f_1} = \tau_{f_2} \circ \tau_{f_1}.$$

Equivalently, under the free-action hypothesis, a representative lift is exactly a pointed flat G-torsor transport system on the protocol category $\mathcal{I}$.

Proof. Let T be a representative lift. For each object $i \in \mathrm{Ob}(\mathcal{I})$, the protocol specifies an orbit

$$\overline{R}(i) \in \mathsf{Phys}.$$

Its fibre

$$\pi^{-1}(\overline{R}(i))$$

is a G-torsor under the given action because, by hypothesis, the orbit fibre is a free G-orbit. For each morphism $f : i \to j$, write

$$T(f) = (g_f, T(i)).$$

Then left multiplication by g_f defines a map

$$\tau_f : \pi^{-1}(\overline{R}(i)) \to \pi^{-1}(\overline{R}(j)), \qquad x \mapsto g_f \cdot x.$$

Because $g_f \in G$ acts bijectively on X, the map τ_f is a torsor isomorphism. Functoriality of T gives

$$g_{\mathrm{id}_i} = e \qquad \text{and} \qquad g_{f_2 \circ f_1} = g_{f_2} g_{f_1},$$

hence

$$\tau_{\mathrm{id}_i} = \mathrm{id}, \qquad \tau_{f_2 \circ f_1} = \tau_{f_2} \circ \tau_{f_1}.$$

Thus the transport system is flat. Conversely, suppose given the orbit fibres, chosen representatives $x_i \in \pi^{-1}(\overline{R}(i))$, and a family of torsor isomorphisms τ_f satisfying the displayed flatness relations and the compatibility condition

$$\tau_f(x_i) = x_j$$

for each morphism $f : i \to j$. Because the fibres are free G-torsors, each τ_f is given by left multiplication by a unique element $g_f \in G$, characterized by

$$g_f \cdot x_i = x_j.$$

These elements satisfy

$$g_{\mathrm{id}_i} = e \qquad \text{and} \qquad g_{f_2 \circ f_1} = g_{f_2} g_{f_1}$$

because the τ_f do. Hence the assignments

$$T(i) := x_i, \qquad T(f) := (g_f, x_i)$$

define a functor

$$T : \mathcal{I} \to X /\!\!/ G$$

with

$$\mathrm{pr} \circ T = \overline{R}.$$

Thus the pointed torsor-transport data are equivalent to a representative lift in the free-action setting. □

Remark 5.6.3. The classification in Theorem 5.1.1 already yields a flat transport system of orbit fibres over the protocol category. Under the additional free-action hypothesis of Theorem 5.6.2, those orbit fibres are G-torsors, and the transport system becomes the discrete antecedent of a flat principal G-bundle. After geometric realization of the protocol category, this becomes the usual notion of a flat principal G-bundle. In later chapters this flat transport background first reappears as loop defect and triangle obstruction, and only in the later smooth realization chain is the stabilized obstruction read as curvature.

5.7 Universal property of the action groupoid

The general classification in Theorem 5.1.1 admits a universal reformulation. The action groupoid is the canonical groupoid over the quotient through which every representative reconstruction factors.

Theorem 5.7.1 (Universal property of the action groupoid)**.** *Let* C *be a groupoid, let*

$$\pi_{\mathsf{C}} : \mathsf{C} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a functor, and let

$$\iota : X \to \mathrm{Ob}(\mathsf{C})$$

be a map satisfying

$$\pi_{\mathsf{C}}(\iota(x)) = \pi(x) \qquad \textit{for all } x \in X.$$

Assume moreover that for every $g \in G$ *and every* $x \in X$ *there exists a unique morphism*

$$\eta_{g,x} : \iota(x) \to \iota(g \cdot x)$$

in C *lying over*

$$\mathrm{id}_{\pi(x)} \in \mathsf{Phys}^{\mathrm{disc}},$$

and that these morphisms satisfy

$$\eta_{e,x} = \mathrm{id}_{\iota(x)}, \qquad \eta_{hg,x} = \eta_{h,g\cdot x} \circ \eta_{g,x}.$$

Then there exists a unique functor

$$F : X /\!\!/ G \to \mathsf{C}$$

such that

$$F(x) = \iota(x), \qquad F(g, x) = \eta_{g,x}, \qquad \pi_{\mathsf{C}} \circ F = \mathrm{pr}\,.$$

Proof. Define F on objects by

$$F(x) := \iota(x),$$

and on morphisms by

$$F(g, x) := \eta_{g,x}.$$

This is well defined because (g, x) is a morphism

$$x \to g \cdot x$$

in $X /\!\!/ G$, while $\eta_{g,x}$ is by hypothesis a morphism

$$\iota(x) \to \iota(g \cdot x)$$

in C. For identities,

$$F(\mathrm{id}_x) = F(e, x) = \eta_{e,x} = \mathrm{id}_{\iota(x)} = \mathrm{id}_{F(x)} \,.$$

For composition, let

$$(g, x) : x \to g \cdot x, \qquad (h, g \cdot x) : g \cdot x \to hg \cdot x.$$

Then

$$F\big((h, g \cdot x) \circ (g, x)\big) = F(hg, x) = \eta_{hg,x}.$$

By the cocycle condition,

$$\eta_{hg,x} = \eta_{h,g\cdot x} \circ \eta_{g,x} = F(h, g \cdot x) \circ F(g, x).$$

Hence F preserves composition. Next, for objects,

$$(\pi_{\mathsf{C}} \circ F)(x) = \pi_{\mathsf{C}}(\iota(x)) = \pi(x) = \mathrm{pr}(x),$$

and for morphisms,

$$(\pi_{\mathsf{C}} \circ F)(g, x) = \pi_{\mathsf{C}}(\eta_{g,x}) = \mathrm{id}_{\pi(x)} = \mathrm{pr}(g, x).$$

Thus

$$\pi_{\mathsf{C}} \circ F = \mathrm{pr} \,.$$

Uniqueness is immediate, because every object of $X /\!\!/ G$ is an element $x \in X$, and every morphism is of the form (g, x). A functor with the stated properties is therefore determined to take exactly the values defined above. □

Remark 5.7.2. Once $\pi : X \to \mathsf{Phys}$ is fixed, the action groupoid $X /\!\!/ G$ is the canonical groupoid carrying all representative reconstructions compatible with the G-action. Every other such reconstruction factors uniquely through it.

5.8 Sections and the object locus

Proposition 5.8.1 (Sections and global representative choice)**.** *Giving a representative choice for every orbit in* Phys *is equivalent to specifying a section*

$$s : \mathsf{Phys} \to X$$

of the orbit projection π*.*

Proof. A section $s : \mathsf{Phys} \to X$ satisfies

$$\pi(s(p)) = p \qquad \text{for every } p \in \mathsf{Phys}.$$

Thus s assigns to each orbit p a representative

$$s(p) \in \pi^{-1}(p).$$

This is exactly a global representative choice. Conversely, suppose one has chosen, for each $p \in \mathsf{Phys}$, an element

$$x_p \in \pi^{-1}(p).$$

Then the assignment

$$s(p) := x_p$$

defines a map $s : \mathsf{Phys} \to X$ satisfying

$$\pi(s(p)) = p.$$

Hence s is a section of π. □

Remark 5.8.2. The object locus contains neither more nor less than preferred representatives in orbit fibres.

5.9 Holonomy and the morphism locus

Definition 5.9.1 (Transport labels and holonomy)**.** Let

$$T : \mathcal{I} \to X /\!\!/ G$$

be a representative lift. For each morphism $f : i \to j$, write

$$T(f) = (g_f, T(i)).$$

If

$$\gamma = f_n \circ \cdots \circ f_1$$

is a composable path in $\mathcal{I}$, define

$$g_\gamma := g_{f_n} \cdots g_{f_1} \in G.$$

If γ is a loop at i, the element g_γ is called the *holonomy* of γ.

Lemma 5.9.2 (Functoriality of path labels). *For composable paths γ_1, γ_2 in $\mathcal{I}$, one has*

$$g_{\gamma_2 \circ \gamma_1} = g_{\gamma_2} g_{\gamma_1}.$$

Proof. Because T is a functor,

$$T(\gamma_2 \circ \gamma_1) = T(\gamma_2) \circ T(\gamma_1).$$

Write

$$T(\gamma_1) = (g_{\gamma_1}, T(i)), \qquad T(\gamma_2) = (g_{\gamma_2}, T(j)).$$

Then

$$T(\gamma_2) \circ T(\gamma_1) = (g_{\gamma_2}, T(j)) \circ (g_{\gamma_1}, T(i)) = (g_{\gamma_2} g_{\gamma_1}, T(i)).$$

By uniqueness of the group label in the action groupoid, this is exactly

$$T(\gamma_2 \circ \gamma_1) = (g_{\gamma_2 \circ \gamma_1}, T(i)).$$

Hence

$$g_{\gamma_2 \circ \gamma_1} = g_{\gamma_2} g_{\gamma_1}.$$

□

Definition 5.9.3 (Gauge modification). Let

$$T : \mathcal{I} \to X /\!\!/ G$$

be a representative lift, and let $(h_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ be a family of elements of G. The *gauge modification* of T by (h_i) is the functor

$$T^h : \mathcal{I} \to X /\!\!/ G$$

defined on objects by

$$T^h(i) := h_i \cdot T(i)$$

and on morphisms by

$$T^h(f) := \big(h_j g_f h_i^{-1},\ h_i \cdot T(i)\big) \qquad (f : i \to j).$$

Lemma 5.9.4. *The assignment $T \mapsto T^h$ defines a representative lift of the same quotient protocol $\overline{R}$.*

Proof. Since $\pi(h_i \cdot T(i)) = \pi(T(i))$, the object assignments remain over the same orbit classes. For a morphism $f : i \to j$,

$$(h_j g_f h_i^{-1}) \cdot (h_i \cdot T(i)) = h_j(g_f \cdot T(i)) = h_j \cdot T(j) = T^h(j),$$

so the stated arrow is well defined in $X /\!\!/ G$. Identities and composition follow from the corresponding identities in G:

$$h_i e h_i^{-1} = e, \qquad (h_k g_2 h_j^{-1})(h_j g_1 h_i^{-1}) = h_k (g_2 g_1) h_i^{-1}.$$

Thus T^h is a functor. Finally,

$$\mathrm{pr}(T^h(i)) = \pi(T(i)) = \overline{R}(i),$$

so $\mathrm{pr} \circ T^h = \overline{R}$. □

Theorem 5.9.5 (Trivial holonomy and reduction to the object locus)**.** *Suppose $\mathcal{I}$ is a connected groupoid and let*

$$T : \mathcal{I} \to X /\!/ G$$

be a representative lift of

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}.$$

Then the following are equivalent.

(i) *Every loop in $\mathcal{I}$ has trivial holonomy.*

(ii) *There exists a gauge-equivalent representative lift*

$$T' : \mathcal{I} \to X /\!/ G$$

with identity morphism labels on every morphism.

(iii) *The morphism-level enrichment of the protocol is globally trivializable, so that, up to gauge, only object-level representative choice remains.*

Proof. (i)⇒(ii). Fix a base object $i_0 \in \mathrm{Ob}(\mathcal{I})$. For each object $i \in \mathrm{Ob}(\mathcal{I})$, choose a path

$$\gamma_i : i_0 \to i$$

in the connected groupoid $\mathcal{I}$ and define

$$h_i := g_{\gamma_i} \in G.$$

Because every loop has trivial holonomy, h_i is independent of the chosen path. Indeed, if γ_i' is another path from i_0 to i, then $(\gamma_i')^{-1} \circ \gamma_i$ is a loop at i_0 in $\mathcal{I}$, so

$$e = g_{(\gamma_i')^{-1} \circ \gamma_i} = g_{\gamma_i'}^{-1} g_{\gamma_i},$$

hence $g_{\gamma_i} = g_{\gamma_i'}$. Consider the gauge modification $T^{h^{-1}}$ corresponding to the family h_i^{-1}. For a morphism $f : i \to j$, the new label is

$$h_j^{-1} g_f h_i.$$

Since γ_j may be chosen as $f \circ \gamma_i$, connectedness of the groupoid and path-independence give

$$h_j = g_f h_i.$$

Therefore

$$h_j^{-1} g_f h_i = e.$$

Thus every morphism of $T^{h^{-1}}$ has identity label. (ii)⇒(iii). If a gauge-equivalent lift has identity labels on every morphism, then all transport data are trivial. Only the representatives on objects remain. Thus the morphism locus reduces to the object locus. (iii)⇒(i). If the morphism-level enrichment is globally trivializable, then after passing to a trivialized lift every morphism has label e. Hence every path has label e, and therefore every loop has trivial holonomy. Since gauge conjugation preserves triviality of loop labels, the original lift also has trivial holonomy on every loop. □

Remark 5.9.6. The point is not that a literal global section $\mathsf{Phys} \to X$ must exist on all of Phys. The point relevant here is sharper: under the same pathwise hypotheses used in the theorem above, vanishing holonomy is exactly the condition under which the morphism locus disappears, up to gauge, and the lift reduces to representative choice.

5.10 Rectangular completeness as the minimal primitive

Theorem 5.10.1 (Minimal primitive for canonical lift classification)**.** *Rectangular comparison completeness is necessary and sufficient for the action-groupoid lift classification developed above to be canonically determined from the comparison datum* $(U, \mathcal{C})$*.*

Proof. Under the closure clause item (SP3), assume rectangular completeness. By the classification theorem at the comparison layer, the canonical factor map

$$\Theta : U \to X_A \times X_B$$

is bijective. Hence U is canonically identified with

$$X := X_A \times X_B.$$

By the diagonal action theorem, the intrinsic symmetry group

$$G = \mathrm{Aut}(U, \mathcal{C})$$

acts canonically and diagonally on X, and the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G$$

is therefore canonically defined. Consequently the action groupoid

$$X /\!\!/ G,$$

the component functor

$$\mathrm{pr} : X /\!\!/ G \to \mathsf{Phys}^{\mathrm{disc}},$$

and the representative lifting problem itself are all canonically determined from $(U, \mathcal{C})$. Conversely, suppose rectangular completeness fails. Then the canonical factor map Θ is not bijective. Therefore no canonical product presentation

$$U \cong X_A \times X_B$$

is available. Without such a canonical carrier X, there is no canonical diagonal action of $\mathrm{Aut}(U, \mathcal{C})$ on a product space, hence no canonical orbit projection $\pi : X \to \mathsf{Phys}$, and therefore no canonical action groupoid $X /\!\!/ G$. In particular, the notions of representative lift, section, transport label, and holonomy are then not intrinsic consequences of the primitive comparison data alone. Thus rectangular completeness is both sufficient and necessary for the universal representative-lift classification to be canonically defined. □

Corollary 5.10.2 (Unique minimal primitive)**.** *Rectangular comparison completeness is the unique minimal primitive at the comparison layer determining canonical diagonal redundancy and hence the object–transport dichotomy.*

Proof. By Theorem 5.10.1, rectangular completeness is necessary and sufficient for the canonical construction of the carrier X, the diagonal action of G, the quotient map π, and the action groupoid $X /\!\!/ G$. All subsequent lifting data depend on this canonical package and are therefore available exactly under rectangular completeness. Hence rectangular completeness is the unique minimal primitive determining the two-locus classification. □

5.11 Conclusion

The structural chain proved in this chapter is

$$(U, \mathcal{C}) \Longrightarrow U \cong X_A \times X_B \Longrightarrow G \curvearrowright X \Longrightarrow \pi : X \to \mathsf{Phys}$$
$$\Longrightarrow \mathrm{pr} : X /\!\!/ G \to \mathsf{Phys}^{\mathrm{disc}} \Longrightarrow \text{universal lift classification.}$$

At the representative level, every enrichment of quotient semantics is exhausted by the following two loci, which may occur together:

(i) representative choice on objects, equivalently the object locus;

(ii) transport and holonomy on morphisms, equivalently the morphism locus.

No third locus exists. This chapter therefore isolates the categorical source of the two-locus principle that governs the remainder of the stack. The object locus reappears later as representative selection and residue. The morphism locus reappears as transport, route dependence, the first finite triangle obstruction, the later loop-defect globalization, and finally curvature after smooth realization. This classification is the exact point at which quotient semantics passes into transport algebra. Accordingly, the no-third-locus statement is not heuristic but complete: every admissible enrichment factor enters through at least one of the two classified loci, and possibly both, namely representative selection on objects and transport data on morphisms.

Chapter 6 turns this classification into explicit calculus, constructing lifts and transport cocycles in fully concrete form.

Chapter 6

Action Groupoid Lifts and the Two Loci of Enrichment

6.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the action-groupoid lift calculus implementing the quotient-descend and transport-visibility clauses (items (SP4) and (SP5)). Building on the locus classification of chapter 5, it develops the explicit lift calculus used in chapters 7 and 8. Chapter 5 established that representative enrichment is exhausted by representative choice and morphism-level transport, possibly with both and with no third locus. Once diagonal redundancy determines the canonical quotient

$$\pi : X \to \mathsf{Phys} := X/G,$$

every quotient-level protocol

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

admits representative realizations precisely as lifts through the action groupoid

$$X /\!\!/ G.$$

The goal of the present chapter is to give a complete algebraic classification of such lifts and to identify the precise transport structure they carry. The first main result is the classification theorem: every representative lift is given by object-level representatives together with morphism-level transport data. The second main result is the gauge principle for such lifts. Changing representatives by pointwise group elements conjugates the transport cocycle in the expected way. Thus representative choice is noncanonical, whereas loop holonomy survives up to conjugacy and furnishes the invariant content of the morphism locus. The third main result is the structural strengthening that matters for the later obstruction theory. The transport cocycle of a lift is not merely a book-keeping device. It is already a flat G-connection on the protocol category, in the strict categorical sense that after groupoid completion it becomes a functor

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG,$$

where BG is the one-object groupoid with automorphism group G. Equivalently, a representative lift is exactly a representative choice together with a flat G-connection whose restriction to the original protocol arrows carries the chosen representatives compatibly from source to target. This strengthening is decisive for the later chapters. The present chapter does not yet produce the later smooth obstruction carriers. It fixes the flat transport background whose first finite failure is isolated in chapter 7, globalized into loop-based descent obstruction in chapter 8, identified with that finite witness in chapter 9, and then carried through observer-level irreversible descent and stitching in chapters 10 and 11 before the later smooth/interface chapters expose the quadratic carrier and curvature realization. In particular, the later triangle obstruction is to be read as the first nontrivial defect beyond this flat compositional regime. Accordingly, the logical spine is not merely

$$\text{quotient protocol} \Longrightarrow \text{lift} \Longrightarrow \text{transport},$$

but rather

$$\begin{gathered}\text{quotient protocol} \Longrightarrow \text{lift} \Longrightarrow \text{compatible flat } G\text{-transport} \\ \Longrightarrow \text{first finite triangle witness} \Longrightarrow \text{loop-obstruction globalization} \\ \Longrightarrow \text{bridge identification} \Longrightarrow \text{observer-level irreversible descent} \\ \Longrightarrow \text{stitching continuation} \Longrightarrow \text{quadratic carrier} \Longrightarrow \text{curvature realization.}\end{gathered}$$

Throughout the chapter all arguments are purely algebraic and categorical. No topology, geometry, or smooth structure is assumed. All assumptions below are local protocol conditions (existence of lifts, connectedness, or gauge choices), not independent foundational axioms.

6.2 Closed systems and quotient semantics

The minimal ambient structure required for the lifting problem is isolated as follows.

Definition 6.2.1 (Closed system with diagonal redundancy). A closed system with diagonal redundancy consists of

$$(X, G, \pi)$$

where

- X is a nonempty set,
- G is a group acting on X on the left,

- $$\pi : X \to X/G$$

  is the orbit projection.

One writes

$$\mathsf{Phys} := X/G.$$

Remark 6.2.2. In the applications treated in chapters 2 and 3 one has

$$X = X_A \times X_B, \qquad G = \mathrm{Aut}(U, \mathcal{C}), \qquad \mathsf{Phys} = X/G.$$

However no product structure on X is used in this chapter. Only the group action and its quotient are required.

Definition 6.2.3 (Admissible report)**.** A report

$$R : X \to S$$

is *admissible* if it is invariant under the G-action:

$$R(g \cdot x) = R(x) \qquad \forall g \in G,\ x \in X.$$

Lemma 6.2.4 (Canonical descent)**.** *A report $R : X \to S$ is admissible if and only if it factors uniquely through the orbit projection:*

$$R = \widetilde{R} \circ \pi$$

for a unique map

$$\widetilde{R} : \mathsf{Phys} \to S.$$

Proof. If R is admissible define

$$\widetilde{R}([x]) := R(x).$$

If $[x] = [y]$ then $y = g \cdot x$ for some $g \in G$, hence $R(y) = R(x)$, proving well-definedness. Conversely if $R = \widetilde{R} \circ \pi$ then

$$R(g \cdot x) = \widetilde{R}(\pi(g \cdot x)) = \widetilde{R}(\pi(x)) = R(x).$$

Uniqueness follows from surjectivity of π. □

Remark 6.2.5 (Scope of the chapter)*.* The present chapter does not enlarge the class of admissible reports. Instead it analyzes the structure that appears when one reconstructs representatives realizing quotient-level protocol data.

6.3 The action groupoid

Definition 6.3.1 (Action groupoid (recalled))**.** As in Theorem 5.2.1, the action groupoid

$$X /\!\!/ G$$

is the category defined as follows.

(1) objects are elements $x \in X$,

(2) morphisms $x \to y$ are pairs

$$(g, x)$$

with $g \in G$ satisfying

$$y = g \cdot x,$$

(3) composition is

$$(h, g \cdot x) \circ (g, x) = (hg, x),$$

(4) identities are

$$\mathrm{id}_x = (e, x).$$

Lemma 6.3.2. *$X /\!\!/ G$ is a groupoid.*

Proof. Associativity follows from associativity in G. Each morphism (g, x) has inverse

$$(g^{-1}, g \cdot x).$$

□

Lemma 6.3.3 (Orbit connectivity)**.** *For $x, y \in X$ the following are equivalent.*

(i) *x and y are isomorphic in $X /\!\!/ G$,*

(ii) *$y = g \cdot x$ for some $g \in G$,*

(iii) *$\pi(x) = \pi(y)$.*

Proof. This is the standard characterization of orbit equivalence for group actions. □

Definition 6.3.4 (Discrete quotient category (recalled))**.** As in Theorem 5.3.1, let

$$\mathsf{Phys}^{\mathrm{disc}}$$

denote the discrete category whose objects are the elements of Phys and whose only morphisms are identity morphisms.

Definition 6.3.5 (Canonical quotient functor)**.** Define

$$\mathrm{pr} : X /\!\!/ G \to \mathsf{Phys}^{\mathrm{disc}}$$

by

$$\mathrm{pr}(x) = \pi(x), \qquad \mathrm{pr}(g, x) = \mathrm{id}_{\pi(x)} .$$

Lemma 6.3.6. pr *is a functor.*

Proof. Identities and composition are preserved because $\mathsf{Phys}^{\mathrm{disc}}$ contains only identity morphisms. ☐

Remark 6.3.7. The functor pr forgets all representative and transport data and retains only orbit classes.

6.4 Protocol indexing and the lift existence criterion

Definition 6.4.1 (Protocol indexing category)**.** A *protocol indexing category* is a small category $\mathcal{I}$.

Definition 6.4.2 (Connected components)**.** Let $\pi_0(\mathcal{I})$ denote the set of connected components of the underlying undirected graph of $\mathcal{I}$. Thus two objects $i, j \in \mathrm{Ob}(\mathcal{I})$ lie in the same connected component if and only if there exists a finite zig-zag of morphisms joining them after forgetting orientation.

Definition 6.4.3 (Quotient-level protocol datum)**.** A *quotient-level protocol datum* is a functor

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}.$$

Lemma 6.4.4 (Lift existence criterion)**.** *For a quotient-level protocol datum*

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}},$$

the following are equivalent.

(i) *there exists a lift*

$$T : \mathcal{I} \to X /\!\!/ G \qquad \textit{such that} \qquad \mathrm{pr} \circ T = \overline{R};$$

(ii) $\overline{R}$ *is constant on each connected component of* $\mathcal{I}$*;*

(iii) $\overline{R}$ *factors through the discrete category*

$$\pi_0(\mathcal{I})^{\mathrm{disc}}.$$

Proof. The equivalence of (ii) and (iii) is merely the definition of factorization through the discrete category of connected components. It remains to prove (i)⇔(ii). (i)⇒(ii). Assume a lift

$$T : \mathcal{I} \to X /\!\!/ G$$

exists. Let $i, j \in \mathrm{Ob}(\mathcal{I})$ lie in the same connected component of $\mathcal{I}$. Choose a zig-zag

$$i = i_0 \leftrightsquigarrow i_1 \leftrightsquigarrow \cdots \leftrightsquigarrow i_n = j$$

in the underlying undirected graph. Applying T, each step produces an isomorphism in $X /\!\!/ G$ between the corresponding objects. By repeated use of Theorem 6.3.3,

$$\pi(T(i_k)) = \pi(T(i_{k+1})) \qquad (0 \le k < n).$$

Hence

$$\pi(T(i)) = \pi(T(j)).$$

Since $\mathrm{pr} \circ T = \overline{R}$, one has

$$\overline{R}(i) = \mathrm{pr}(T(i)) = \mathrm{pr}(T(j)) = \overline{R}(j).$$

Thus $\overline{R}$ is constant on each connected component. (ii)⇒(i). Assume $\overline{R}$ is constant on each connected component of $\mathcal{I}$. For each component $C \in \pi_0(\mathcal{I})$, choose an object $i_C \in C$. Since $\pi : X \to \mathsf{Phys}$ is surjective, choose

$$x_C \in X$$

such that

$$\pi(x_C) = \overline{R}(i_C).$$

Define T on objects by

$$T(i) := x_C \qquad \text{for } i \in C.$$

This is well defined because every object lies in a unique connected component. For each morphism $f : i \to j$ with $i, j \in C$, define

$$T(f) := \mathrm{id}_{x_C}.$$

Then T preserves identities and composition because identities do. Finally, if $i \in C$, then

$$(\mathrm{pr} \circ T)(i) = \mathrm{pr}(x_C) = \pi(x_C) = \overline{R}(i_C) = \overline{R}(i),$$

because $\overline{R}$ is constant on C. On morphisms, both sides necessarily take values in identities of the discrete category $\mathsf{Phys}^{\mathrm{disc}}$. Hence

$$\mathrm{pr} \circ T = \overline{R}.$$

So a lift exists. □

Remark 6.4.5 (Meaning of the criterion). Quotient-level protocol data contain no route information. The only obstruction to lifting is componentwise inconsistency of orbit labels. No subtler obstruction appears at this stage.

6.5 Representative choices and transport cocycles

The two layers of data that constitute a lift are isolated explicitly.

Definition 6.5.1 (Representative choice)**.** Let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient-level protocol datum. A *representative choice* over $\overline{R}$ is a function

$$x : \mathrm{Ob}(\mathcal{I}) \to X, \qquad i \mapsto x_i,$$

such that

$$\pi(x_i) = \overline{R}(i) \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I}).$$

Thus a representative choice selects one point of X in each orbit fibre prescribed by the quotient protocol.

Definition 6.5.2 (Transport cocycle)**.** Let x be a representative choice over $\overline{R}$. A *transport cocycle relative to* x is a map

$$g : \mathrm{Mor}(\mathcal{I}) \to G, \qquad f \mapsto g_f,$$

satisfying the following conditions.

(T1) **Target compatibility:** for every morphism $f : i \to j$,

$$g_f \cdot x_i = x_j;$$

(T2) **Identity normalization:** for every object i,

$$g_{\mathrm{id}_i} = e;$$

(T3) **Functorial composition:** for every composable pair

$$i \xrightarrow{f} j \xrightarrow{f'} k,$$

one has

$$g_{f' \circ f} = g_{f'} g_f.$$

Remark 6.5.3. Condition (T1) says that g_f transports the chosen representative at the source of f to the chosen representative at the target of f. Conditions (T2) and (T3) are precisely the identity and composition constraints required for a functor into the action groupoid.

Remark 6.5.4 (Loop defect and holonomy). If $\ell : i \to i$ is an endomorphism in $\mathcal{I}$, then (T1) implies

$$g_\ell \cdot x_i = x_i.$$

Hence

$$g_\ell \in \mathrm{Stab}(x_i) := \{h \in G : h \cdot x_i = x_i\}.$$

Such stabilizer elements are the loop defect, or holonomy, carried by the morphism locus. It first becomes finitely visible as the triangle witness of chapter 7, is globalized into loop-based descent obstruction in chapter 8, identified with that finite witness in chapter 9, and then carried through observer-level irreversible descent and stitching in chapters 10 and 11.

6.6 Classification of lifts

Theorem 6.6.1 (Universal classification of representative lifts)**.** *Under the standing principle Standing Principle 1, let*

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient-level protocol for which lifts exist. Then representative lifts

$$T : \mathcal{I} \to X /\!\!/ G, \qquad \mathrm{pr} \circ T = \overline{R}$$

are in canonical bijection with pairs

$$(x, g)$$

consisting of

(i) *a representative choice*

$$x_i \in X, \qquad \pi(x_i) = \overline{R}(i),$$

(ii) *a transport cocycle*

$$g : \mathrm{Mor}(\mathcal{I}) \to G$$

satisfying

$$g_f \cdot x_i = x_j, \qquad g_{\mathrm{id}_i} = e, \qquad g_{f' \circ f} = g_{f'} g_f.$$

Under this correspondence

$$T(i) = x_i, \qquad T(f) = (g_f, x_i).$$

The possible enrichment data are exhausted by two loci that may occur together: the object layer, consisting of the choice of representatives in the fibres of

$$\pi : X \to \mathsf{Phys},$$

and the morphism layer, consisting of the transport elements of G relating those representatives along arrows. No third locus exists.

The proof of Theorem 6.6.1 proceeds as follows.

Proof. Mutually inverse maps are constructed between the following two sets:

$$(A) := \Big\{ T : \mathcal{I} \to X /\!\!/ G \ : \ \mathrm{pr} \circ T = \overline{R} \Big\},$$

$$(B) := \Big\{ (x, g) \ : \ x \text{ is a representative choice over } \overline{R}, \\ g \text{ is a transport cocycle relative to } x \Big\}.$$

Step 1: from lifts to representative–transport data. Let

$$T : \mathcal{I} \to X /\!\!/ G$$

be a lift of $\overline{R}$, so that

$$\mathrm{pr} \circ T = \overline{R}.$$

For each object $i \in \mathrm{Ob}(\mathcal{I})$, define

$$x_i := T(i) \in X.$$

Then

$$\pi(x_i) = \mathrm{pr}(T(i)) = (\mathrm{pr} \circ T)(i) = \overline{R}(i),$$

so $x = (x_i)$ is a representative choice over $\overline{R}$. Now let $f : i \to j$ be a morphism in $\mathcal{I}$. Since $T(f)$ is a morphism in the action groupoid from x_i to x_j, there exists a unique element $g_f \in G$ such that

$$T(f) = (g_f, x_i) : x_i \to g_f \cdot x_i = x_j.$$

This defines a map

$$g : \mathrm{Mor}(\mathcal{I}) \to G.$$

Condition (T1) is immediate from the target of the morphism $T(f)$:

$$g_f \cdot x_i = x_j.$$

For (T2), functoriality gives

$$T(\mathrm{id}_i) = \mathrm{id}_{T(i)} = \mathrm{id}_{x_i} = (e, x_i).$$

By uniqueness of the group label in an action-groupoid morphism with fixed source, one obtains

$$g_{\mathrm{id}_i} = e.$$

For (T3), let

$$i \xrightarrow{f} j \xrightarrow{f'} k$$

be composable. Because T is a functor,

$$T(f' \circ f) = T(f') \circ T(f).$$

Using the composition law in $X /\!/ G$,

$$(g_{f' \circ f}, x_i) = (g_{f'}, x_j) \circ (g_f, x_i) = (g_{f'} g_f, x_i).$$

Again uniqueness of the group label implies

$$g_{f' \circ f} = g_{f'} g_f.$$

Thus g is a transport cocycle relative to x. This defines a map

$$\Phi : (A) \to (B), \qquad T \mapsto (x, g).$$

Step 2: from representative–transport data to lifts. Conversely, let $(x, g) \in (B)$. Define T on objects by

$$T(i) := x_i.$$

For each morphism $f : i \to j$, define

$$T(f) := (g_f, x_i).$$

By (T1),

$$g_f \cdot x_i = x_j,$$

so $T(f)$ is indeed a morphism

$$x_i \to x_j$$

in the action groupoid. Functoriality is verified. For identities, by (T2),

$$T(\mathrm{id}_i) = (g_{\mathrm{id}_i}, x_i) = (e, x_i) = \mathrm{id}_{x_i} = \mathrm{id}_{T(i)}\,.$$

For composable morphisms

$$i \xrightarrow{f} j \xrightarrow{f'} k,$$

condition (T3) gives

$$T(f' \circ f) = (g_{f' \circ f}, x_i) = (g_{f'} g_f, x_i) = (g_{f'}, x_j) \circ (g_f, x_i) = T(f') \circ T(f).$$

Thus T is a functor

$$T : \mathcal{I} \to X /\!/ G.$$

Finally, because x is a representative choice,

$$\mathrm{pr}(T(i)) = \mathrm{pr}(x_i) = \pi(x_i) = \overline{R}(i)$$

for every object i. On morphisms, both $\mathrm{pr} \circ T$ and $\overline{R}$ necessarily take values in identities of $\mathsf{Phys}^{\mathrm{disc}}$. Hence

$$\mathrm{pr} \circ T = \overline{R}.$$

So T is a lift. This defines a map

$$\Psi : (B) \to (A), \qquad (x, g) \mapsto T.$$

Step 3: the two constructions are inverse. Starting with $T \in (A)$, Step 1 extracts the object values $x_i = T(i)$ and the unique group labels appearing in the morphisms $T(f)$. Reconstruction by Step 2 then returns the same functor T. Conversely, starting with $(x, g) \in (B)$, Step 2 constructs a functor whose object values are exactly the x_i and whose morphism labels are exactly the g_f. Applying Step 1 recovers the original pair (x, g). Thus Φ and Ψ are mutually inverse, yielding the desired canonical bijection. □

Corollary 6.6.2 (Two loci of lift data)**.** *Every representative lift of a quotient protocol is exhausted by the following two loci of data, which may occur together:*

(i) *object-level data: the representative choice x;*

(ii) *morphism-level data: the transport cocycle g.*

No third independent enrichment datum is present.

Proof. This is exactly the content of Theorem 6.6.1. □

Remark 6.6.3. Theorem 6.6.1 is the explicit algebraic form of the two-locus principle proved abstractly in Chapter 5. The object locus is the choice of representatives. The morphism locus is the transport cocycle. All later obstruction theory develops the second locus.

6.7 Gauge modifications

The natural equivalence relation on representative lift data is analyzed as follows.

Definition 6.7.1 (Gauge modification (lift form))**.** As in Theorem 5.9.3, let (x, g) be lift data over $\overline{R}$, and let

$$h = (h_i)_{i \in \mathrm{Ob}(\mathcal{I})}$$

be a family of elements of G. Define new data (x^h, g^h) by

$$x_i^h := h_i \cdot x_i,$$

and for each morphism $f : i \to j$,

$$g_f^h := h_j g_f h_i^{-1}.$$

Lemma 6.7.2 (Gauge modification preserves lift data)**.** *If (x, g) satisfies the transport cocycle conditions, then so does (x^h, g^h).*

Proof. Let $f : i \to j$. Then

$$g_f^h \cdot x_i^h = (h_j g_f h_i^{-1}) \cdot (h_i \cdot x_i) = h_j \cdot (g_f \cdot x_i) = h_j \cdot x_j = x_j^h,$$

so target compatibility holds. For identities,

$$g_{\mathrm{id}_i}^h = h_i g_{\mathrm{id}_i} h_i^{-1} = h_i e h_i^{-1} = e.$$

For composition, if

$$i \xrightarrow{f} j \xrightarrow{f'} k$$

is composable, then

$$g_{f' \circ f}^h = h_k g_{f' \circ f} h_i^{-1} = h_k (g_{f'} g_f) h_i^{-1} = (h_k g_{f'} h_j^{-1})(h_j g_f h_i^{-1}) = g_{f'}^h g_f^h.$$

Thus (x^h, g^h) again satisfies the transport cocycle conditions. $\square$

Proposition 6.7.3 (Gauge-equivalent lifts are naturally isomorphic)**.** *Let (x, g) and (x^h, g^h) be gauge-related lift data, and let*

$$T, T^h : \mathcal{I} \to X /\!\!/ G$$

be the corresponding lifts. Then there is a natural isomorphism

$$T \Longrightarrow T^h.$$

Proof. For each object $i \in \mathrm{Ob}(\mathcal{I})$, define

$$\eta_i := (h_i, x_i) : x_i \to h_i \cdot x_i = x_i^h.$$

Thus

$$\eta_i : T(i) \to T^h(i)$$

is a morphism in the action groupoid. Let $f : i \to j$. Then

$$T(f) = (g_f, x_i), \qquad T^h(f) = (g_f^h, x_i^h).$$

The composite is computed as

$$\eta_j \circ T(f) = (h_j, x_j) \circ (g_f, x_i) = (h_j g_f, x_i),$$

because $x_j = g_f \cdot x_i$. On the other hand,

$$T^h(f) \circ \eta_i = (g_f^h, x_i^h) \circ (h_i, x_i) = (g_f^h h_i, x_i).$$

Since

$$g_f^h h_i = (h_j g_f h_i^{-1}) h_i = h_j g_f,$$

the two composites agree. Hence (η_i) is natural. Each η_i is invertible, with inverse

$$(h_i^{-1}, x_i^h) : x_i^h \to x_i.$$

Therefore T and T^h are naturally isomorphic. $\square$

Remark 6.7.4. Gauge modification changes representatives and conjugates transport. Thus representative choice is noncanonical. The invariant morphism-level residue is holonomy up to conjugacy.

6.8 Path transport and holonomy

Definition 6.8.1 (Path transport)**.** Let (x, g) be representative lift data. For a composable path

$$\gamma = \big(i_0 \xrightarrow{f_1} i_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} i_n\big),$$

define

$$g_\gamma := g_{f_n} \cdots g_{f_1} \in G.$$

Lemma 6.8.2 (Path compatibility)**.** *For every path $\gamma : i \to j$ in $\mathcal{I}$,*

$$g_\gamma \cdot x_i = x_j.$$

Proof. The argument proceeds by induction on the length of the path. For the identity path,

$$g_{\mathrm{id}_i} = e,$$

so

$$g_{\mathrm{id}_i} \cdot x_i = x_i.$$

Suppose the statement holds for a path $\gamma : i \to j$, and let $f : j \to k$ be a morphism. Then

$$g_{f\circ\gamma} = g_f g_\gamma,$$

hence

$$g_{f\circ\gamma} \cdot x_i = g_f \cdot (g_\gamma \cdot x_i) = g_f \cdot x_j = x_k.$$

Thus the claim follows. □

Definition 6.8.3 (Holonomy)**.** If $\ell : i \to i$ is a loop in $\mathcal{I}$, the element

$$g_\ell \in G$$

is called the *holonomy* of ℓ.

Lemma 6.8.4 (Holonomy lies in the stabilizer)**.** *If $\ell : i \to i$ is a loop, then*

$$g_\ell \cdot x_i = x_i.$$

Hence

$$g_\ell \in G_{x_i} := \{g \in G : g \cdot x_i = x_i\}.$$

Proof. This is the case $j = i$ of Theorem 6.8.2. □

Lemma 6.8.5 (Gauge conjugation of path transport)**.** *For every path $\gamma : i \to j$,*

$$g^h_\gamma = h_j g_\gamma h_i^{-1}.$$

Proof. If

$$\gamma = f_n \circ \cdots \circ f_1,$$

then

$$g^h_\gamma = g^h_{f_n} \cdots g^h_{f_1} = (h_j g_{f_n} h^{-1}_{i_{n-1}})(h_{i_{n-1}} g_{f_{n-1}} h^{-1}_{i_{n-2}}) \cdots (h_{i_1} g_{f_1} h^{-1}_i),$$

and all intermediate factors cancel, yielding

$$g^h_\gamma = h_j g_\gamma h^{-1}_i.$$

□

Corollary 6.8.6 (Holonomy conjugation)**.** *For a loop $\ell : i \to i$,*

$$g^h_\ell = h_i g_\ell h^{-1}_i.$$

In particular,

$$g_\ell = e \iff g^h_\ell = e.$$

Proof. This is the special case $i = j$ of Theorem 6.8.5. □

Remark 6.8.7. Loop holonomy is therefore gauge-invariant up to conjugacy. This is the precise invariant carried by the morphism locus before the later obstruction refinements.

6.9 Flat connection interpretation

The classification theorem admits a sharper reformulation than the language of transport cocycles alone suggests. The morphism-level data of a representative lift are exactly the data of a flat G-connection on the protocol category.

Definition 6.9.1 (Classifying groupoid)**.** Let BG denote the one-object groupoid whose automorphism group is G.

Definition 6.9.2 (Groupoid completion)**.** Let $\mathcal{G}(\mathcal{I})$ denote the groupoid completion of $\mathcal{I}$, obtained by freely adjoining inverses to all morphisms.

Lemma 6.9.3 (Transport cocycles extend uniquely to the groupoid completion)**.** *Let (x, g) be representative lift data on $\mathcal{I}$. Then the transport cocycle g extends uniquely to a functor*

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG.$$

Proof. Condition

$$g_{f' \circ f} = g_{f'} g_f$$

gives functoriality on $\mathcal{I}$. Since BG is a groupoid, every element of G is invertible. Hence every formally adjoined inverse in $\mathcal{G}(\mathcal{I})$ must be sent to the corresponding inverse element of G. This determines a unique extension. □

Remark 6.9.4 (Loop holonomy after completion). The extended functor

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG$$

assigns transport labels not only to directed paths in $\mathcal{I}$ but to arbitrary zig-zags obtained by formally inverting morphisms. Accordingly, holonomy of the associated flat G-connection is most naturally measured on loops in the groupoid completion $\mathcal{G}(\mathcal{I})$, not merely on endomorphisms already present in $\mathcal{I}$.

Definition 6.9.5 (Flat G-connection on a protocol category)**.** A *flat G-connection* on the protocol category $\mathcal{I}$ is a functor

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG.$$

Remark 6.9.6 (Why the connection is flat). The term *flat* is determined by strict compositionality. Transport along a composite path is exactly the product of the transports along its pieces:

$$g_{f' \circ f} = g_{f'} g_f.$$

No additional defect term appears at this stage. Thus the transport system carries strict compositional closure on the protocol category itself. The residual invariant visible at this stage is loop holonomy. Later, in the local connected-protocol setup of Theorem 6.10.1, trivial holonomy is exactly the condition under which that morphism-level residue can be gauged away.

Theorem 6.9.7 (Representative lifts as representatives plus compatible flat G-connections)**.** *Under the standing principle Standing Principle 1, let*

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient protocol. Then representative lifts of $\overline{R}$ are equivalent to pairs consisting of

(i) *a representative choice*

$$x_i \in X \qquad \textit{with} \qquad \pi(x_i) = \overline{R}(i) \quad (i \in \mathrm{Ob}(\mathcal{I})),$$

(ii) *a flat G-connection on $\mathcal{I}$, equivalently a functor*

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG,$$

whose restriction to each morphism $f : i \to j$ of $\mathcal{I}$ satisfies

$$\widehat{g}(f) \cdot x_i = x_j.$$

Equivalently, every representative lift determines, and is determined by,

$$\textit{representatives} \quad + \quad \textit{compatible flat } G\textit{-connection.}$$

Proof. By Theorem 6.6.1, representative lifts are equivalent to pairs (x, g) consisting of a representative choice and a transport cocycle, so in particular

$$g_f \cdot x_i = x_j$$

for every morphism $f : i \to j$ of $\mathcal{I}$. By Theorem 6.9.3, such a transport cocycle g extends uniquely to a functor

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG.$$

Its restriction to $\mathcal{I}$ is the original cocycle g. By Theorem 6.9.5, such a functor is exactly a flat G-connection on the protocol category. Therefore representative lifts are equivalent to representative choices together with flat G-connections that remain compatible with those representatives on the original protocol arrows. □

Remark 6.9.8 (Relation with flat principal bundles). At the strict level proved here, the safest formulation is the categorical one:

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG.$$

After geometric realization of the nerve of $\mathcal{I}$, the same data become the usual notion of a flat principal G-bundle. Thus the present chapter produces the discrete categorical antecedent of the later geometric transport theory.

Remark 6.9.9 (Bridge to the triangle obstruction). This chapter does not yet produce the later smooth obstruction carriers. It fixes the flat transport background whose first finite failure is isolated in chapter 7, globalized into loop-based descent obstruction in chapter 8, identified with that finite witness in chapter 9, and then carried through observer-level irreversible descent and stitching in chapters 10 and 11 before the later smooth/interface chapters expose the quadratic carrier and curvature realization.

6.10 Gauge fixing and flat transport

Flatness of the extended connection is characterized as gauge triviality.

Theorem 6.10.1 (Flat transport and gauge trivialization). *In the local connected-protocol setup, assume that $\mathcal{I}$ is connected, and let*

$$\widehat{g} : \mathcal{G}(\mathcal{I}) \to BG$$

be the flat G-connection associated to representative lift data (x, g) by Theorem 6.9.3. Then the following are equivalent.

(i) *every loop in the groupoid completion has trivial holonomy:*

$$\widehat{g}(\omega) = e \qquad \textit{for every loop } \omega : i \to i \textit{ in } \mathcal{G}(\mathcal{I});$$

(ii) *there exists a gauge modification* $h = (h_i)$ *such that every transport label in the groupoid completion becomes trivial:*

$$\widehat{g}^h(\eta) = e \qquad \textit{for every } \eta \in \mathrm{Mor}(\mathcal{G}(\mathcal{I}));$$

(iii) (x, g) *is gauge-equivalent to lift data with trivial transport cocycle on* $\mathcal{I}$.

Proof. (ii)⇒(i). If every transport label in the gauge h is trivial on $\mathcal{G}(\mathcal{I})$, then in particular

$$\widehat{g}^h(\omega) = e$$

for every loop ω in $\mathcal{G}(\mathcal{I})$. Since gauge transformation acts by conjugation at the endpoints, triviality of a loop label is gauge-invariant. Hence

$$\widehat{g}(\omega) = e$$

for every loop ω. (i)⇒(ii). Fix a base object $i_0 \in \mathrm{Ob}(\mathcal{I})$. Since $\mathcal{I}$ is connected, for each object i choose a morphism

$$\gamma_i : i_0 \to i$$

in the groupoid completion $\mathcal{G}(\mathcal{I})$. Set

$$k_i := \widehat{g}(\gamma_i), \qquad h_i := k_i^{-1}.$$

The independence of k_i from the chosen path is first shown. If γ_i' is another morphism $i_0 \to i$ in $\mathcal{G}(\mathcal{I})$, then

$$\omega := \gamma_i'^{-1} \circ \gamma_i$$

is a loop at i_0. By hypothesis,

$$\widehat{g}(\omega) = e.$$

By functoriality,

$$e = \widehat{g}(\gamma_i'^{-1} \circ \gamma_i) = \widehat{g}(\gamma_i')^{-1}\widehat{g}(\gamma_i),$$

hence

$$\widehat{g}(\gamma_i') = \widehat{g}(\gamma_i).$$

Thus k_i is well defined. Now let

$$\eta : i \to j$$

be any morphism in $\mathcal{G}(\mathcal{I})$. Then

$$\gamma_j^{-1} \circ \eta \circ \gamma_i$$

is a loop at i_0, so again by hypothesis,

$$\widehat{g}(\gamma_j^{-1} \circ \eta \circ \gamma_i) = e.$$

Using functoriality,

$$\widehat{g}(\gamma_j)^{-1}\widehat{g}(\eta)\widehat{g}(\gamma_i) = e,$$

and therefore

$$\widehat{g}(\eta) = \widehat{g}(\gamma_j)\widehat{g}(\gamma_i)^{-1} = k_j k_i^{-1}.$$

After gauge modification by $h_i = k_i^{-1}$, the transformed transport label on η is

$$\widehat{g}^{\,h}(\eta) = h_j\,\widehat{g}(\eta)\,h_i^{-1} = k_j^{-1}(k_j k_i^{-1})k_i = e.$$

Thus the entire connection becomes trivial on $\mathcal{G}(\mathcal{I})$. (ii)⇒(iii). Restrict the trivialized connection on $\mathcal{G}(\mathcal{I})$ to the original category $\mathcal{I}$. Then every transport label on $\mathcal{I}$ is e, so the original lift data are gauge-equivalent to lift data with trivial transport cocycle. (iii)⇒(ii). A trivial transport cocycle on $\mathcal{I}$ extends uniquely to the trivial functor on $\mathcal{G}(\mathcal{I})$. Hence the extended connection is trivial on all morphisms of the groupoid completion. □

Corollary 6.10.2 (Flatness isolates object-level enrichment)**.** *In the local connected-protocol setup, assume that $\mathcal{I}$ is connected. Then the only obstruction to reducing a representative lift to pure object-level representative choice is nontrivial loop holonomy of the extended flat G-connection on $\mathcal{G}(\mathcal{I})$.*

Proof. This is exactly Theorem 6.10.1. □

6.11 Structural meaning for later chapters

Remark 6.11.1 (Match with the later obstruction chapters). This chapter converts the abstract two-locus principle into explicit algebraic data. The object locus becomes representative choice. The morphism locus becomes transport, encoded by a G-valued cocycle and measured invariantly by holonomy. Later chapters do not introduce a new kind of structure. They refine this same morphism locus through route dependence, the first finite triangle obstruction, the later loop-defect globalization, graded commutator defect, and finally curvature after smooth realization.

Remark 6.11.2 (Conceptual summary). The logical output of the chapter is not merely

$$\text{quotient protocol} \quad + \quad \text{lift} \quad = \quad \text{representatives} \quad + \quad \text{transport},$$

but more precisely

$$\text{quotient protocol} \quad + \quad \text{lift} \quad = \quad \text{representatives} \quad + \quad \text{compatible flat } G\text{-connection}.$$

Equivalently,

$$\text{lift through } X /\!\!/ G \quad = \quad \text{object locus} \quad + \quad \text{morphism locus},$$

where the object locus is representative choice and the morphism locus is a flat G-connection on the protocol category whose restriction to the original protocol arrows carries the chosen representatives compatibly from source to target. This is the exact algebraic seed from which the later transport-obstruction chain grows: chapter 7 first isolates its finite witness, chapters 8 and 9 globalize and identify the same residue, chapters 10 and 11 carry it through observer-level irreversible descent and stitching, and only later do chapters 12 and 13 expose the quadratic carrier and curvature realization. The next obstruction is therefore not a new primitive. It is the first nontrivial defect beyond the flat transport regime isolated here.

6.12 Conclusion

This chapter converts the two-locus theorem into explicit algebra: representative lifts are encoded by object choices together with a flat G-connection whose restriction to the original protocol arrows carries the chosen representatives compatibly from source to target. In the local connected-protocol setup, the sole obstruction to reducing this data to pure object-level representative choice is nontrivial loop holonomy.

Accordingly, chapter 7 isolates the first finite witness of this morphism defect, chapter 8 globalizes that same witness into loop-based descent obstruction, chapters 9 to 11 carry it through bridge identification, observer-level irreversible descent, and stitching, and only afterward does it feed the later smooth realization chapters chapters 12 and 13.

Chapter 7

Comparison Triangles and the Multi–Point Boundary

7.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the first finite visible transport obstruction in the sense of item (SP5). Proceeding from the lift classification, it identifies the first finite triangle boundary that underlies chapters 8 and 9. Chapters 5 and 6 established that representative enrichment is exhausted by representative choice and morphism-level transport, possibly with both and with no third locus; equivalently, every representative lift is given by representative choice together with transport data, and no third layer of enrichment is present (Theorems 6.6.1 and 6.6.2). In particular, once a quotient-level protocol is lifted to representatives in X, the only possible obstruction to global compatibility lies in the transport data. The present chapter identifies the first nontrivial finite regime in which that obstruction becomes intrinsically visible. At the one-point level there is no issue: quotient semantics is complete, and admissible reports are exactly the G-invariant maps

$$R : X \to S,$$

equivalently those that factor through the orbit projection

$$\pi : X \to \mathsf{Phys} := X/G$$

(Theorem 6.2.4). At the multi-point level, however, a stronger compatibility problem appears. Several representatives may each be individually compatible with the quotient data, yet fail to admit a *single common diagonal gauge* aligning them simultaneously. This failure is the first genuinely relational boundary beyond ordinary quotient semantics. The chapter has three aims.

1. We formulate diagonal n-point semantics and show that n-point coherent observables separate exactly the diagonal G-orbits in X^n.

2. We prove a stabilizer–coset criterion for simultaneous diagonal alignment. This converts the problem into a concrete intersection problem inside G.

3. We define the *minimal detection size*

$$m = \min\{|I| : \bigcap_{i \in I} C_i = \varnothing\},$$

prove the associated minimality theorem, and isolate the regime $m = 3$, called the *triangle regime*. In that regime, pairwise compatibility still holds but global compatibility fails, so triangles are the first nontrivial finite witnesses of transport obstruction.

This chapter is entirely algebraic. No topology, smoothness, metric, probability, or dynamics is assumed.

7.2 Diagonal n-point semantics

Let G act diagonally on X^n by

$$g \cdot (x_1, \ldots, x_n) := (g \cdot x_1, \ldots, g \cdot x_n).$$

Definition 7.2.1 (Diagonal orbit projection)**.** For each $n \geq 1$, let

$$\pi_n : X^n \to X^n/G$$

denote the orbit projection for the diagonal action.

Definition 7.2.2 (Diagonal equivalence)**.** For $x, y \in X^n$, write

$$x \sim_\Delta y$$

if there exists $g \in G$ such that

$$y = g \cdot x.$$

Equivalently,

$$x \sim_\Delta y \iff \pi_n(x) = \pi_n(y).$$

Definition 7.2.3 (n-point coherent observable)**.** Let S be a set. A map

$$R^{(n)} : X^n \to S$$

is called *n-point coherent* if it is invariant under the diagonal action of G, i.e.

$$R^{(n)}(g \cdot x) = R^{(n)}(x) \qquad \text{for all } g \in G,\ x \in X^n.$$

Theorem 7.2.4 (Diagonal separation)**.** *For $x, y \in X^n$, the following are equivalent.*

1.

$$\pi_n(x) = \pi_n(y).$$

2. *Every n-point coherent observable takes the same value on x and y.*

Proof. Assume first that $\pi_n(x) = \pi_n(y)$. Then $y = g \cdot x$ for some $g \in G$. Hence for every n-point coherent observable $R^{(n)}$,

$$R^{(n)}(y) = R^{(n)}(g \cdot x) = R^{(n)}(x).$$

Conversely, the quotient map

$$\pi_n : X^n \to X^n/G$$

is itself invariant under the diagonal action. Therefore, if every diagonally invariant map agrees on x and y, then in particular

$$\pi_n(x) = \pi_n(y).$$

□

Remark 7.2.5 (Multi-point boundary). For $n = 1$, Theorem 7.2.4 is just ordinary quotient semantics. For $n \geq 2$, diagonal orbit structure on X^n may contain strictly more information than the pointwise orbit list in Phys^n, because diagonal equivalence requires one common element of G for all coordinates simultaneously.

7.3 Alignment and stabilizer cosets

We now convert simultaneous diagonal alignment into an intersection problem in the group G.

Definition 7.3.1 (Stabilizers and alignment cosets)**.** Fix $n \geq 1$, points

$$x_1, \ldots, x_n \in X,$$

and elements

$$g_1, \ldots, g_n \in G.$$

Define

$$y_k := g_k^{-1} \cdot x_k \qquad (1 \leq k \leq n).$$

Let

$$H_k := \mathrm{Stab}(x_k) = \{h \in G : \ h \cdot x_k = x_k\}.$$

Define the associated alignment coset

$$C_k := H_k g_k^{-1} \subseteq G.$$

Lemma 7.3.2 (Coset intersection criterion)**.** *With notation as in Theorem 7.3.1,*

$$(y_1, \dots, y_n) \sim_\Delta (x_1, \dots, x_n) \iff \bigcap_{k=1}^{n} C_k \neq \varnothing.$$

Proof. By definition,

$$(y_1, \dots, y_n) \sim_\Delta (x_1, \dots, x_n)$$

if and only if there exists $a \in G$ such that

$$a \cdot x_k = y_k \qquad \text{for all } k.$$

Using $y_k = g_k^{-1} \cdot x_k$, this is equivalent to

$$a \cdot x_k = g_k^{-1} \cdot x_k \qquad \text{for all } k.$$

Multiplying on the left by g_k gives

$$g_k a \cdot x_k = x_k \qquad \text{for all } k,$$

hence

$$g_k a \in H_k \qquad \text{for all } k.$$

Equivalently,

$$a \in H_k g_k^{-1} = C_k \qquad \text{for all } k.$$

Thus such an a exists if and only if

$$a \in \bigcap_{k=1}^{n} C_k.$$

□

Remark 7.3.3 (Free-action specialization)*.* If each stabilizer H_k is trivial, then

$$C_k = \{g_k^{-1}\},$$

and Theorem 7.3.2 reduces to the statement that simultaneous diagonal alignment holds if and only if

$$g_1 = \cdots = g_n.$$

Thus in the free regime the multi-point obstruction is exactly the failure of all transport labels to coincide.

7.4 Minimal detection size

Definition 7.4.1 (Minimal detection size)**.** With notation as in Theorem 7.3.1, define

$$m := \min \left\{ |I| : I \subseteq \{1, \ldots, n\}, \bigcap_{i \in I} C_i = \varnothing \right\},$$

with the convention $m = \infty$ if no such subset exists.

Thus m is the smallest number of points whose simultaneous diagonal alignment fails.

Remark 7.4.2 (Restriction notation). If $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$, we write

$$(x_i)_{i \in I} := (x_{i_1}, \ldots, x_{i_r}), \qquad (y_i)_{i \in I} := (y_{i_1}, \ldots, y_{i_r}),$$

in the induced order on I.

Theorem 7.4.3 (Minimal obstruction theorem)**.** *In the local alignment setup, assume $m < \infty$. Then:*

1. *For every $I \subseteq \{1, \ldots, n\}$ with $|I| < m$,*

$$(y_i)_{i \in I} \sim_\Delta (x_i)_{i \in I}.$$

2. *There exists $I \subseteq \{1, \ldots, n\}$ with $|I| = m$ such that*

$$(y_i)_{i \in I} \not\sim_\Delta (x_i)_{i \in I}.$$

Hence m is the unique minimal cardinality at which failure of a common diagonal gauge can be detected.

Proof. Let $I \subseteq \{1, \ldots, n\}$ with $|I| < m$. By minimality of m, one has

$$\bigcap_{i \in I} C_i \neq \varnothing.$$

Applying Theorem 7.3.2 to the restricted tuples gives

$$(y_i)_{i \in I} \sim_\Delta (x_i)_{i \in I}.$$

This proves the first statement. By definition of m, there exists I with $|I| = m$ and

$$\bigcap_{i \in I} C_i = \varnothing.$$

Applying Theorem 7.3.2 again yields

$$(y_i)_{i \in I} \not\sim_\Delta (x_i)_{i \in I}.$$

This proves the second statement. □

Corollary 7.4.4 ($m = 2$ and $m = 3$ regimes)**.** *With notation as in Theorem 7.3.1:*

1. $m = 2$ *if and only if there exist* $i \neq j$ *such that*

$$C_i \cap C_j = \varnothing.$$

2. $m = 3$ *if and only if every pairwise intersection is nonempty, but there exist distinct* i, j, k *such that*

$$C_i \cap C_j \cap C_k = \varnothing.$$

Proof. This is immediate from the definition of m. □

7.5 The triangle regime

Definition 7.5.1 (Triangle regime)**.** We say that the configuration is in the *triangle regime* if

$$m = 3.$$

Theorem 7.5.2 (Strict minimality of triangles)**.** *In the local alignment setup, assume the triangle regime. Then there exist distinct indices* i, j, k *such that:*

1. *every two-point restriction among* $\{i, j, k\}$ *is diagonally alignable;*

2. *the three-point restriction*

$$(y_i, y_j, y_k)$$

is not diagonally alignable with

$$(x_i, x_j, x_k).$$

Consequently:

1. *no two-point coherent observable detects the obstruction;*

2. *some three-point coherent observable does detect it.*

Proof. By definition of the triangle regime, there exists a triple $\{i, j, k\}$ such that

$$C_i \cap C_j \cap C_k = \varnothing.$$

Again because $m = 3$, every subset of cardinality < 3 has nonempty intersection. In particular,

$$C_i \cap C_j \neq \varnothing, \qquad C_i \cap C_k \neq \varnothing, \qquad C_j \cap C_k \neq \varnothing.$$

By Theorem 7.3.2, each corresponding two-point restriction is diagonally equivalent. This proves the first statement. Applying Theorem 7.3.2 to the triple gives

$$(y_i, y_j, y_k) \not\sim_\Delta (x_i, x_j, x_k).$$

This proves the second statement. The detectability statements follow from Theorem 7.2.4: the two-point restrictions are diagonally equivalent and therefore cannot be separated by any two-point coherent observable, whereas the three-point restriction is not diagonally equivalent and is therefore separated by some three-point coherent observable. The separating three-point observable may be taken to be the orbit projection

$$\pi_3 : X^3 \to X^3/G.$$

□

Corollary 7.5.3 (Triangle detectability)**.** *In the triangle regime, $n = 3$ is strictly minimal for detecting the obstruction to global diagonal alignment.*

Proof. This is just a restatement of Theorem 7.5.2. □

Remark 7.5.4 (Structural meaning of the triangle regime). The significance of the triangle regime is not that every obstruction must already occur at size 3. The significance is that $m = 3$ is the first regime in which global incompatibility can persist despite complete pairwise compatibility. Thus triangles are the first nontrivial finite witnesses of genuinely relational transport failure.

7.6 Transport realization from lifts

We now connect the coset formalism of Theorems 7.3.1 and 7.3.2 directly to the lift and transport structure of chapter 6, without introducing any auxiliary base object or path choice.

Definition 7.6.1 (Finite transport sample)**.** Let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\text{disc}}$$

be a quotient-level protocol, and let

$$T : \mathcal{I} \to X /\!\!/ G$$

be a representative lift. A *finite transport sample* extracted from T consists of:

1. a finite family of objects
$$i_1, \dots, i_n \in \mathrm{Ob}(\mathcal{I}),$$
2. representatives
$$x_k := T(i_k) \in X \qquad (1 \le k \le n),$$
3. morphisms in the action groupoid
$$(g_k, x_k) : x_k \to y_k \qquad (1 \le k \le n),$$
arising from the transport data carried by T.

For such a sample, define

$$H_k := \mathrm{Stab}(x_k), \qquad C_k := H_k g_k^{-1}.$$

Lemma 7.6.2 (Intrinsicity of the induced cosets)**.** *Let a finite transport sample be extracted from a representative lift T. Then the associated cosets*

$$C_k = H_k g_k^{-1}$$

depend only on the transport morphisms of the lift and are unchanged by replacing any chosen transport label g_k by another label representing the same morphism with the same source x_k. Consequently all intersection properties of the family $\{C_k\}$ are intrinsic to the lift.

Proof. Fix k. In the action groupoid, a morphism with source x_k and target y_k may be written as

$$(g_k, x_k) : x_k \to y_k.$$

Suppose the same morphism is represented by another group element $g_k' \in G$ with

$$(g_k', x_k) : x_k \to y_k.$$

Then

$$g_k \cdot x_k = y_k = g_k' \cdot x_k,$$

hence

$$g_k' g_k^{-1} \cdot x_k = x_k.$$

Therefore

$$g_k' g_k^{-1} \in H_k.$$

Write

$$g_k' = h_k g_k \qquad \text{for some } h_k \in H_k.$$

Then

$$H_k (g_k')^{-1} = H_k g_k^{-1} h_k^{-1} = H_k g_k^{-1},$$

because $h_k^{-1} \in H_k$. Thus the coset C_k is unchanged. Since this holds for each k, all intersections of the family $\{C_k\}$ are unchanged as well. □

Proposition 7.6.3 (Transport induces alignment cosets)**.** *Every finite transport sample extracted from a representative lift determines canonically a family of alignment cosets*

$$C_k = H_k g_k^{-1}, \qquad H_k = \mathrm{Stab}(x_k),$$

and the associated alignment problem is exactly the intrinsic alignment problem of Theorem 7.3.2.

Proof. This is immediate from Theorem 7.6.1, Theorem 7.6.2, and Theorem 7.3.2. □

Corollary 7.6.4 (Triangle obstruction for transport)**.** *If the cosets induced by a finite transport sample lie in the triangle regime, then every pair of transported representatives is simultaneously alignable, but some triple is not. Thus the first nontrivial finite obstruction to global transport alignment is triangular.*

Proof. Immediate from Theorem 7.5.2. □

Remark 7.6.5 (Morphism-locus character of the obstruction)*.* The obstruction detected here does not arise from quotient semantics itself. It arises only after representatives have been chosen and transport labels compared across several points. Thus the triangle regime lies strictly in the morphism locus of enrichment, i.e. strictly beyond the object-level representative data isolated in Theorems 6.6.2 and 6.9.7.

7.7 Interpretation

The multi-point obstruction studied here is the first place where the morphism-locus transport data of chapter 6 become visible as an intrinsically relational phenomenon. At one point, quotient semantics forgets all representative dependence. At two points, there may or may not already be an obstruction. But in the triangle regime, every pair remains compatible while the triple fails. This is the first configuration in which pairwise transport coherence ceases to imply global coherence. The conclusion is exact: the triangle regime is the first regime in which compatibility ceases to be controlled by lower-arity data. In that sense, triangles are the first irreducible finite carriers of global transport incompatibility. This is the precise algebraic precursor of the later geometric picture. The next chapter globalizes this triangular incompatibility into a loop-based descent obstruction, and chapter 9 then identifies that loop defect and the triangle witness as two presentations of the same intrinsic morphism-locus invariant. Under the subsequent loop-level globalization of chapter 8 and the triangle–loop identification of chapter 9, this same finite witness becomes the corresponding relational loop defect. Under smooth realization, the infinitesimal form of that loop defect is curvature.

7.8 Conclusion

The finite obstruction boundary is now sharp: the triangle regime is the first irreducible configuration in which transport data are pairwise compatible yet globally inconsistent. No lower-arity test detects this failure.

Accordingly, chapter 8 globalizes this finite witness into a loop-representation descent obstruction, and chapter 9 then identifies the finite triangle witness and loop defect as two presentations of the same morphism-level obstruction datum.

Chapter 8

Descent–Obstruction Classification

8.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter globalizes the transport-visibility clause item (SP5) from finite triangle witnesses to loop-level obstruction representations. Starting from the first finite witness isolated in chapter 7, it develops the global obstruction classification via loop representations required in chapter 9. Chapter 7 identified the first finite witness of failure of a common diagonal gauge (Theorem 7.5.2). The present chapter addresses the complementary global problem: how does one classify all representative lifts of a connected quotient protocol up to gauge? The answer is controlled by a tree–loop decomposition. Fix a spanning tree in the underlying undirected graph of $\mathcal{I}$. After a suitable gauge normalization, all representatives become constant and every morphism in the tree groupoid carries trivial transport. All residual transport then lives on loops. More precisely, the remaining datum is a homomorphism from the based loop group of the groupoid completion

$$\Omega_{i_0}(\mathcal{G}(\mathcal{I}))$$

into the stabilizer subgroup of a chosen base representative. This loop representation is the intrinsic obstruction to descent. Tree transport carries no gauge-invariant content; loop transport is the residue. Endpoint-determined transport on the groupoid completion is equivalent to triviality of this residue. The chapter proves three things.

1. Every lift is gauge-equivalent to a tree-normalized lift with constant representative choice.

2. Tree-normalized lifts are completely determined by their based loop representation.

3. Gauge-equivalence classes of lifts are classified by conjugacy classes of homomorphisms
$$\rho : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*, \qquad H_* = \mathrm{Stab}(x_*),$$
where x_* is the chosen base representative.

Thus the morphism locus of the two-locus principle admits a complete and explicit obstruction theory. Beyond the already-separated object-level representative data, no additional morphism-level obstruction parameter appears.

8.2 Connected quotient protocols and their lifts

Let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a quotient-level protocol, where $\mathcal{I}$ is a connected small category and $\mathsf{Phys}^{\mathrm{disc}}$ is the discrete category on the quotient state space Phys. By Theorem 6.4.4, connectedness of $\mathcal{I}$ implies that $\overline{R}$ is constant. We therefore fix the unique point

$$p \in \mathsf{Phys}$$

such that

$$\overline{R}(i) = p \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I}).$$

Choose once and for all a representative

$$x_* \in X \qquad \text{with} \qquad \pi(x_*) = p,$$

and set

$$H_* := \mathrm{Stab}(x_*) = \{h \in G : \; h \cdot x_* = x_*\}.$$

We work with representative lift data (x, g) in the sense of Theorem 6.6.1: thus

$$x_i \in X \qquad (i \in \mathrm{Ob}(\mathcal{I})),$$

and

$$g_f \in G \qquad (f \in \mathrm{Mor}(\mathcal{I})),$$

satisfy

$$g_f \cdot x_i = x_j \qquad \text{for every } f : i \to j, \tag{8.2.1}$$

$$g_{\mathrm{id}_i} = e \qquad \text{for every } i, \tag{8.2.2}$$

$$g_{f' \circ f} = g_{f'} g_f \qquad \text{for every composable } i \xrightarrow{f} j \xrightarrow{f'} k. \tag{8.2.3}$$

Equivalently, these data determine a functor

$$T : \mathcal{I} \to X /\!\!/ G$$

lifting $\overline{R}$.

8.3 Tree normalization

We now isolate the gauge-removable part of the transport.

Definition 8.3.1 (Underlying graph and spanning tree)**.** Let $|\mathcal{I}|$ denote the underlying undirected graph of $\mathcal{I}$: its vertices are the objects of $\mathcal{I}$, and an undirected edge joins i and j whenever $\mathcal{I}$ contains a morphism $i \to j$ or $j \to i$. A *spanning tree* $\mathcal{T} \subseteq |\mathcal{I}|$ is a connected, acyclic spanning subgraph.

Fix a base object

$$i_0 \in \mathrm{Ob}(\mathcal{I})$$

and a spanning tree $\mathcal{T} \subseteq |\mathcal{I}|$. Let

$$\mathcal{G}(\mathcal{I})$$

denote the groupoid completion of $\mathcal{I}$, as in Theorem 6.9.2. Let

$$\mathcal{G}(\mathcal{T}) \subseteq \mathcal{G}(\mathcal{I})$$

denote the subgroupoid generated by the tree edges. Since $\mathcal{T}$ is a tree, for each object i there is a unique morphism

$$\gamma_i : i_0 \to i$$

in $\mathcal{G}(\mathcal{T})$.

Lemma 8.3.2 (Tree-path transport)**.** *For every object i,*

$$g_{\gamma_i} \cdot x_{i_0} = x_i.$$

Proof. By Theorem 6.9.3, the transport cocycle extends uniquely from $\mathcal{I}$ to a functor

$$\widetilde{T} : \mathcal{G}(\mathcal{I}) \to X /\!\!/ G.$$

Write

$$\widetilde{T}(\gamma_i) = (g_{\gamma_i}, x_{i_0}).$$

Since $\gamma_i : i_0 \to i$, the target of this morphism is x_i. By definition of the action groupoid,

$$g_{\gamma_i} \cdot x_{i_0} = x_i.$$

□

Theorem 8.3.3 (Tree normalization)**.** *There exists a gauge modification $h = (h_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ such that the gauge-modified lift datum (x^h, g^h) satisfies:*

1. $$x_i^h = x_* \qquad \textit{for all } i \in \mathrm{Ob}(\mathcal{I});$$

2. *for every morphism τ in the tree groupoid $\mathcal{G}(\mathcal{T})$,*

$$g_\tau^h = e.$$

In particular, every tree edge carries trivial transport after normalization.

Proof. Because $\overline{R}$ is constant with value p, every representative x_i lies in the orbit $\pi^{-1}(p)$. In particular,

$$\pi(x_{i_0}) = \pi(x_*) = p.$$

Hence there exists $u \in G$ such that

$$u \cdot x_{i_0} = x_*.$$

By Theorem 8.3.2,

$$g_{\gamma_i} \cdot x_{i_0} = x_i \qquad \text{for all } i.$$

Define

$$h_i := u\, g_{\gamma_i}^{-1} \qquad (i \in \mathrm{Ob}(\mathcal{I})).$$

We first compute the transformed representatives. By definition of gauge modification,

$$x_i^h = h_i \cdot x_i = u\, g_{\gamma_i}^{-1} \cdot x_i.$$

Using $x_i = g_{\gamma_i} \cdot x_{i_0}$, we obtain

$$x_i^h = u\, g_{\gamma_i}^{-1} \cdot (g_{\gamma_i} \cdot x_{i_0}) = u \cdot x_{i_0} = x_*.$$

Thus all transformed representatives are equal to the base representative x_*. Next let $\tau : i \to j$ be any morphism in the tree groupoid $\mathcal{G}(\mathcal{T})$. Then in $\mathcal{G}(\mathcal{T})$ one has

$$\tau = \gamma_j \circ \gamma_i^{-1}.$$

Hence, by functoriality of the extended transport,

$$g_\tau = g_{\gamma_j} g_{\gamma_i}^{-1}.$$

The transformed transport label is therefore

$$g_\tau^h = h_j g_\tau h_i^{-1} = (u g_{\gamma_j}^{-1})(g_{\gamma_j} g_{\gamma_i}^{-1})(g_{\gamma_i} u^{-1}) = e.$$

This proves the second statement. □

Definition 8.3.4 (Tree-normalized lift)**.** A representative lift datum (x, g) is called *tree-normalized* if

$$x_i = x_* \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I}),$$

and

$$g_\tau = e \qquad \text{for all } \tau \in \mathrm{Mor}(\mathcal{G}(\mathcal{T})).$$

Remark 8.3.5 (What remains after normalization)*.* After tree normalization, the object-level data are constant and the entire tree transport vanishes. All residual information is therefore concentrated on loops. This is the descent–obstruction form of the morphism locus.

8.4 Loop representations

Let

$$\Omega_{i_0}(\mathcal{G}(\mathcal{I})) := \mathrm{Hom}_{\mathcal{G}(\mathcal{I})}(i_0, i_0)$$

denote the based loop group at i_0, with group law given by composition in $\mathcal{G}(\mathcal{I})$.

Lemma 8.4.1 (Loop holonomy lands in the stabilizer)**.** *If (x, g) is tree-normalized, then for every loop*

$$\ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$$

one has

$$g_\ell \in H_* = \mathrm{Stab}(x_*).$$

Proof. Since (x, g) is tree-normalized, the representative at the base object is x_*. Because ℓ is a loop at i_0, the corresponding morphism in the action groupoid is

$$(g_\ell, x_*) : x_* \to x_*.$$

Thus

$$g_\ell \cdot x_* = x_*,$$

which is exactly the statement that $g_\ell \in H_*$. □

Definition 8.4.2 (Obstruction representation)**.** For a tree-normalized lift datum (x, g), define

$$\rho_g : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*, \qquad \rho_g(\ell) := g_\ell.$$

Lemma 8.4.3. *The map ρ_g is a group homomorphism.*

Proof. Let $\ell_1, \ell_2 \in \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$. Since transport is functorial on the groupoid completion,

$$g_{\ell_2 \circ \ell_1} = g_{\ell_2} g_{\ell_1}.$$

Therefore

$$\rho_g(\ell_2 \circ \ell_1) = g_{\ell_2 \circ \ell_1} = g_{\ell_2} g_{\ell_1} = \rho_g(\ell_2)\rho_g(\ell_1).$$

Thus ρ_g is a homomorphism. □

Lemma 8.4.4 (Gauge conjugation of the loop representation)**.** *Let (x, g) be tree-normalized, and let $k \in H_*$. Define a gauge transformation by*

$$h_i := k \qquad \textit{for all } i \in \mathrm{Ob}(\mathcal{I}).$$

Then the transformed tree-normalized lift datum satisfies

$$\rho_{g^h}(\ell) = k\, \rho_g(\ell)\, k^{-1} \qquad \textit{for all } \ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I})).$$

Proof. For any morphism $\alpha : i \to j$ in $\mathcal{G}(\mathcal{I})$, the gauge transformation rule gives

$$g^h_\alpha = h_j g_\alpha h_i^{-1} = k g_\alpha k^{-1}.$$

Applying this to a loop $\ell : i_0 \to i_0$ yields

$$\rho_{g^h}(\ell) = g^h_\ell = k g_\ell k^{-1} = k\, \rho_g(\ell)\, k^{-1}.$$

□

8.5 Reconstruction from the loop representation

We now prove the converse statement: a homomorphism into the base stabilizer determines a unique tree-normalized lift.

Definition 8.5.1 (Loop reduction map)**.** For any morphism

$$\alpha : i \to j$$

in $\mathcal{G}(\mathcal{I})$, define the associated based loop

$$\lambda(\alpha) := \gamma_j^{-1} \circ \alpha \circ \gamma_i \in \Omega_{i_0}(\mathcal{G}(\mathcal{I})).$$

Lemma 8.5.2 (Multiplicativity of loop reduction)**.** *For composable morphisms*

$$i \xrightarrow{\alpha} j \xrightarrow{\beta} k$$

in $\mathcal{G}(\mathcal{I})$,

$$\lambda(\beta \circ \alpha) = \lambda(\beta) \circ \lambda(\alpha).$$

Proof. By definition,

$$\lambda(\beta \circ \alpha) = \gamma_k^{-1} \circ \beta \circ \alpha \circ \gamma_i.$$

Insert the identity

$$\mathrm{id}_j = \gamma_j \circ \gamma_j^{-1}$$

between β and α. Then

$$\lambda(\beta \circ \alpha) = (\gamma_k^{-1} \circ \beta \circ \gamma_j) \circ (\gamma_j^{-1} \circ \alpha \circ \gamma_i) = \lambda(\beta) \circ \lambda(\alpha).$$

□

Proposition 8.5.3 (Reconstruction from a loop representation)**.** *Let*

$$\rho : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*$$

be a homomorphism. Then there exists a unique tree-normalized representative lift datum (x, g) *such that*

$$\rho_g = \rho.$$

Proof. Define

$$x_i := x_* \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I}).$$

For any morphism $\alpha : i \to j$ in $\mathcal{G}(\mathcal{I})$, define

$$g_\alpha := \rho(\lambda(\alpha)) \in H_* \subseteq G.$$

We first check that this determines a functor

$$\widetilde{T}_\rho : \mathcal{G}(\mathcal{I}) \to X /\!\!/ G.$$

Because $g_\alpha \in H_* = \mathrm{Stab}(x_*)$, one has

$$g_\alpha \cdot x_i = g_\alpha \cdot x_* = x_* = x_j.$$

Hence

$$(g_\alpha, x_*) : x_i \to x_j$$

is a well-defined morphism in the action groupoid. For the identity morphism id_i, one has

$$\lambda(\mathrm{id}_i) = \gamma_i^{-1} \circ \mathrm{id}_i \circ \gamma_i = \mathrm{id}_{i_0},$$

so

$$g_{\mathrm{id}_i} = \rho(\mathrm{id}_{i_0}) = e.$$

For composable α, β, Theorem 8.5.2 and the homomorphism property of ρ give

$$g_{\beta\circ\alpha} = \rho(\lambda(\beta \circ \alpha)) = \rho(\lambda(\beta) \circ \lambda(\alpha)) = \rho(\lambda(\beta))\rho(\lambda(\alpha)) = g_\beta g_\alpha.$$

Thus $\widetilde{T}_\rho$ is a functor. Restricting from $\mathcal{G}(\mathcal{I})$ to $\mathcal{I}$ gives a representative lift datum (x, g). It remains to verify tree-normalization. If $\tau : i \to j$ is a morphism in the tree groupoid $\mathcal{G}(\mathcal{T})$, then

$$\tau = \gamma_j \circ \gamma_i^{-1},$$

hence

$$\lambda(\tau) = \gamma_j^{-1} \circ \tau \circ \gamma_i = \gamma_j^{-1} \circ \gamma_j \circ \gamma_i^{-1} \circ \gamma_i = \mathrm{id}_{i_0}.$$

Therefore

$$g_\tau = \rho(\mathrm{id}_{i_0}) = e.$$

So the lift is tree-normalized. Finally, for a loop $\ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$,

$$\lambda(\ell) = \gamma_{i_0}^{-1} \circ \ell \circ \gamma_{i_0} = \ell,$$

because $\gamma_{i_0} = \mathrm{id}_{i_0}$. Hence

$$\rho_g(\ell) = g_\ell = \rho(\ell).$$

Thus $\rho_g = \rho$. Uniqueness is immediate: a tree-normalized lift is determined by its values on all morphisms, and for each α those values are determined to be $\rho(\lambda(\alpha))$. □

8.6 Descent–obstruction classification

We now combine normalization, reconstruction, and gauge conjugation into the main theorem.

Theorem 8.6.1 (Descent–obstruction classification)**.** *Under the standing principle Standing Principle 1, let $\mathcal{I}$ be connected, fix a base object i_0, and fix a representative*

$$x_* \in \pi^{-1}(p), \qquad H_* = \mathrm{Stab}(x_*).$$

Then gauge-equivalence classes of representative lifts of the connected quotient protocol

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

are classified by H_*-conjugacy classes of homomorphisms

$$\rho : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*.$$

Equivalently: after tree normalization, all residual transport data are recorded exactly by the based loop representation into the base stabilizer.

Proof. We divide the proof into four steps. *Step 1: every gauge class admits a tree-normalized representative.* This is Theorem 8.3.3. Hence every gauge-equivalence class contains at least one tree-normalized lift datum. *Step 2: every tree-normalized lift determines a homomorphism $\rho : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*$.* Given a tree-normalized lift datum (x, g), define ρ_g by Theorem 8.4.2. By Theorem 8.4.1,

$$\rho_g(\ell) \in H_* \qquad \text{for all } \ell,$$

and by Theorem 8.4.3, ρ_g is a homomorphism. *Step 3: every such homomorphism reconstructs a unique tree-normalized lift.* This is exactly Theorem 8.5.3. Therefore tree-normalized lifts are in bijection with homomorphisms

$$\rho : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*.$$

Step 4: residual gauge acts by H_*-conjugation. Let (x, g) and (x', g') be two tree-normalized lifts belonging to the same gauge-equivalence class. Since both are tree-normalized, all representatives are equal to x_*, and all tree morphisms have trivial transport. Let $h = (h_i)$ be a gauge modification with

$$(x', g') = (x^h, g^h).$$

Because

$$x'_i = x_i = x_* \qquad \text{for all } i,$$

one has

$$h_i \cdot x_* = x_*,$$

hence each $h_i \in H_*$. Now let $\tau : i \to j$ be a tree morphism. Since both lifts are tree-normalized,

$$e = g'_\tau = h_j g_\tau h_i^{-1} = h_j e h_i^{-1},$$

so

$$h_j = h_i.$$

As the tree is connected, it follows that all h_i are equal to a single element

$$k \in H_*.$$

By Theorem 8.4.4,

$$\rho_{g'}(\ell) = k\,\rho_g(\ell)\,k^{-1} \qquad \text{for all } \ell.$$

Thus gauge-equivalent tree-normalized lifts determine H_*-conjugate homomorphisms. Conversely, let $\rho, \rho' : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*$ be two homomorphisms that are H_*-conjugate, say

$$\rho'(\ell) = k\,\rho(\ell)\,k^{-1} \qquad (k \in H_*).$$

Let (x, g) and (x', g') be the corresponding tree-normalized lifts given by Theorem 8.5.3. Then the constant gauge transformation $h_i := k$ transforms (x, g) into (x', g'). Indeed, for any $\alpha : i \to j$,

$$g^h_\alpha = k\,g_\alpha\,k^{-1} = k\,\rho(\lambda(\alpha))\,k^{-1} = \rho'(\lambda(\alpha)) = g'_\alpha.$$

Since both reconstructed lifts are tree-normalized, this proves that they are gauge-equivalent. Combining the four steps yields the classification. □

8.7 Endpoint-determined transport

The obstruction representation measures exactly the failure of endpoint determinacy. In this chapter, endpoint-determined transport means endpoint-determined transport on the groupoid completion $\mathcal{G}(\mathcal{I})$.

Corollary 8.7.1 (Endpoint-determined transport)**.** *For a connected indexing category, the following are equivalent.*

1. *Transport on the groupoid completion $\mathcal{G}(\mathcal{I})$ is endpoint-determined.*
2. *Every loop holonomy is trivial.*
3. *The obstruction representation*

   $$\rho_g : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*$$

   is the trivial homomorphism.

Proof. We first prove (i)⇒(ii). Assume transport is endpoint-determined on $\mathcal{G}(\mathcal{I})$. Let

$$\ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$$

be any loop at i_0. Since ℓ and id_{i_0} have the same source and target, endpoint-determinacy implies

$$g_\ell = g_{\mathrm{id}_{i_0}} = e.$$

Thus every loop holonomy is trivial. Next, (ii)⇒(iii) is immediate from the definition of ρ_g: if every loop holonomy is trivial, then

$$\rho_g(\ell) = g_\ell = e \qquad \text{for every } \ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I})),$$

so ρ_g is the trivial homomorphism. Finally, we prove (iii)⇒(i). Assume ρ_g is trivial. Let

$$\alpha, \beta : i \to j$$

be morphisms in $\mathcal{G}(\mathcal{I})$ with the same source and target. Then

$$\omega := \beta^{-1} \circ \alpha$$

is a loop at i. Conjugating by the tree paths gives a based loop at i_0,

$$\lambda(\omega) = \gamma_i^{-1} \circ \omega \circ \gamma_i \in \Omega_{i_0}(\mathcal{G}(\mathcal{I})).$$

Since ρ_g is trivial,

$$g_{\lambda(\omega)} = e.$$

By definition of λ,

$$g_{\lambda(\omega)} = g_{\gamma_i}^{-1} g_\omega g_{\gamma_i},$$

hence

$$g_\omega = e.$$

Using $\omega = \beta^{-1} \circ \alpha$, functoriality gives

$$e = g_\omega = g_{\beta^{-1} \circ \alpha} = g_\beta^{-1} g_\alpha.$$

Therefore

$$g_\alpha = g_\beta.$$

Thus transport on $\mathcal{G}(\mathcal{I})$ is endpoint-determined. □

Corollary 8.7.2 (Flatness isolates object-level enrichment)**.** *A connected representative lift is gauge-equivalent to one with trivial transport if and only if its obstruction representation is trivial.*

Proof. Assume first that the obstruction representation is trivial. By Theorem 8.7.1, transport is endpoint-determined on $\mathcal{G}(\mathcal{I})$. Let (x', g') be a tree-normalized representative of the gauge class, given by Theorem 8.3.3. Since (x', g') is tree-normalized, one has

$$g'_\tau = e \qquad \text{for every } \tau \in \mathrm{Mor}(\mathcal{G}(\mathcal{T})).$$

Now let $\alpha : i \to j$ be any morphism in $\mathcal{G}(\mathcal{I})$. The tree morphism

$$\tau := \gamma_j \circ \gamma_i^{-1} : i \to j$$

has the same source and target as α. Since transport is endpoint-determined,

$$g'_\alpha = g'_\tau = e.$$

Hence every morphism of $\mathcal{G}(\mathcal{I})$, and therefore every morphism of $\mathcal{I}$, carries trivial transport in the tree-normalized representative. Thus the original lift is gauge-equivalent to one with trivial transport. Conversely, if the lift is gauge-equivalent to one with trivial transport, then in that representative

$$g_\ell = e \qquad \text{for every } \ell \in \Omega_{i_0}(\mathcal{G}(\mathcal{I})),$$

so the obstruction representation is trivial. Since triviality is preserved under gauge conjugation, the original lift also has trivial obstruction representation. □

8.8 Interpretation

The classification theorem shows that the morphism locus of enrichment has a rigid internal structure. Tree transport carries no gauge-invariant content. The residual content is carried entirely by the loop representation into the stabilizer of a single representative. Thus the obstruction is not, in general, a single group element, but a conjugacy class of homomorphisms

$$\Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*.$$

This is the exact descent–obstruction form of the two-locus spine:

- the object locus is representative choice;
- the morphism locus is the obstruction representation.

No third enrichment appears, because the action groupoid contains only objects in X and morphisms labeled by elements of G. This loop representation is the algebraic residue that persists into the later chapters. In chapter 9 it is identified with the finite triangle witness on the minimal directed triangle subsystem, from there it drives the observer-level irreversible descent of chapter 10, continues through the stitched refinement/observer arena of chapter 11, and only later does the obstruction chain pass into graded commutator defects and curvature after smooth realization.

8.9 Conclusion

The loop-side classification is complete: after tree normalization, all gauge-invariant morphism enrichment is concentrated in the conjugacy class of the obstruction representation $\Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to H_*$. This identifies descent failure as the exact global residue of transport nontriviality.

Accordingly, chapter 9 proves that this global loop obstruction and the finite triangle witness are two presentations of one invariant datum, and chapter 10 then derives observer-level irreversible descent from that same static transport obstruction.

Chapter 9

Triangle–Loop Identification as the Rigidity Bridge

9.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter identifies the finite and loop forms of item (SP5) as one and the same obstruction datum. Closing the discrete obstruction arc, it proves that triangle failure and loop-defect failure represent the same datum, preparing chapter 10; only later, through the observer-level and stitching continuation, does this bridge feed the smooth realization chapters. Chapters 7 and 8 isolated two a priori different descriptions of nontrivial morphism-level enrichment. From chapter 7, the first finite obstruction to a common diagonal gauge occurs at three points. In the triangle regime, every pairwise restriction is diagonally alignable, but the full triple is not. Equivalently, the corresponding stabilizer cosets have nonempty pairwise intersections while the full triple intersection is empty. From chapter 8, transport on a connected protocol, after tree normalization, is classified by a based loop representation into a stabilizer group. In that language, nontriviality is carried by loop defect. The purpose of the present chapter is to prove that, on the minimal three-vertex subsystem carrying the first finite obstruction, these are the same phenomenon. The triangle obstruction is not merely analogous to loop defect. It is exactly the obstruction to descending transport through triangle coherence. The key point is categorical. In the free path groupoid, a directed triangle is merely a loop. The relevant new structure appears only after one specifies that directed triangle loops are to count as coherent. This produces a quotient groupoid. The correct obstruction is then not nontrivial transport *on* that quotient, but rather the failure of the original transport functor to factor through it. The logical output of the chapter is the following equivalence on a directed triangle subsystem:

$$\begin{aligned}
\text{triangle regime} &\iff \text{failure of coherent vertex gauge}\\
&\iff [\kappa_\ell] \neq [e]\\
&\iff \text{failure of descent through triangle coherence}
\end{aligned}$$

Here $[\kappa_\ell]$ denotes the conjugacy class of the triangle defect in the stabilizer group. The passage to conjugacy is essential: the defect element itself depends on normalization choices, but its triviality or nontriviality does not. No appeal is made in this chapter to smooth realization, curvature, filtrations, or quadratic carriers. Its role is narrower and more rigid. It identifies the first finite diagonal obstruction with the first obstruction to transport descent through triangle coherence. **Dependence on labeled results.** The chapter uses:

- representative lift classification and tree normalization (Theorems 6.6.1 and 8.3.3),
- the loop-based formulation of transport in the groupoid completion (chapter 8),
- the strict minimality of the triangle regime (Theorem 7.5.2).

Use in subsequent chapters. The chapter provides:

- a canonical obstruction class carried by directed triangles,
- identification of the minimal triangle subsystem as the first finite witness of failure of transport descent through triangle coherence,
- the same static obstruction class carried next into the observer-level irreversible descent of chapter 10.

9.2 Single-object transport extracted from a connected lift

We first isolate the exact single-object transport regime determined by the classification theory of chapter 8. Let

$$\overline{R} : \mathcal{I} \to \mathsf{Phys}^{\mathrm{disc}}$$

be a connected quotient protocol, and let

$$(x, g)$$

be representative lift data in the sense of Theorem 6.6.1. Fix a base representative

$$x_* \in X$$

lying over the unique value of $\overline{R}$, and set

$$H_* := \mathrm{Stab}(x_*).$$

By Theorem 8.3.3, every gauge class admits a tree-normalized representative. We therefore fix, throughout the chapter, a tree-normalized lift datum

$$(x, g)$$

such that

$$x_i = x_* \qquad \text{for all } i \in \mathrm{Ob}(\mathcal{I}),$$

and

$$g_\tau = e \qquad \text{for all } \tau \in \mathrm{Mor}(\mathcal{G}(\mathcal{T}))$$

for a chosen spanning tree $\mathcal{T} \subseteq |\mathcal{I}|$.

Lemma 9.2.1 (Single-object transport values). *For every morphism*

$$\alpha : i \to j \qquad \textit{in } \mathcal{G}(\mathcal{I}),$$

one has

$$g_\alpha \in H_*.$$

Proof. Since the lift is tree-normalized,

$$x_i = x_j = x_* \qquad \text{for all } i, j.$$

Because the transport law on the groupoid completion satisfies

$$g_\alpha \cdot x_i = x_j,$$

it follows that

$$g_\alpha \cdot x_* = x_*.$$

Thus $g_\alpha \in \mathrm{Stab}(x_*) = H_*$. □

Definition 9.2.2 (Finite transport subsystem). A *finite transport subsystem* of the tree-normalized lift is a finite directed graph

$$\mathcal{N} = (V, E)$$

together with a graph morphism into the underlying directed graph of $\mathcal{G}(\mathcal{I})$. Equivalently, for each directed edge

$$e : u \to v \qquad (u, v \in V),$$

one specifies a morphism

$$\alpha_e : u \to v \qquad \text{in } \mathcal{G}(\mathcal{I}).$$

The associated transport assignment is the map

$$\tau : E \to H_*, \qquad \tau(e) := g_{\alpha_e}.$$

Remark 9.2.3 (Single-object comparison regime). By Theorem 9.2.1, every finite transport subsystem is governed by a one-object comparison groupoid with automorphism group H_*. All representative data have been gauged to the single object x_*, and the entire morphism locus is carried by the edge-labeling map

$$\tau : E \to H_*.$$

This is the precise single-object transport regime used below.

9.3 Free path groupoid and triangle coherence

Let

$$\mathcal{N} = (V, E)$$

be a finite transport subsystem with transport assignment

$$\tau : E \to H_*.$$

Definition 9.3.1 (Free path groupoid (recalled)). Let

$$\mathrm{Path}^{\pm}(\mathcal{N})$$

denote the free path groupoid on $\mathcal{N}$, exactly as in Theorem 3.8.1: it is obtained by adjoining to each directed edge

$$e : u \to v$$

a formal inverse

$$e^{-1} : v \to u,$$

and imposing only the groupoid relations generated by identities and cancellation of adjacent inverse pairs.

Definition 9.3.2 (Extended transport functor). The edge-labeling map $\tau : E \to H_*$ extends uniquely to a functor

$$\mathrm{Hol} : \mathrm{Path}^{\pm}(\mathcal{N}) \to BH_*,$$

where BH_* denotes the one-object groupoid with automorphism group H_*. On formal inverses,

$$\mathrm{Hol}(e^{-1}) = \mathrm{Hol}(e)^{-1} = \tau(e)^{-1}.$$

Definition 9.3.3 (Directed triangle loop). A *directed triangle loop* in $\mathcal{N}$ is a composable loop of the form

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big).$$

Definition 9.3.4 (Triangle defect). For a directed triangle loop

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big),$$

its *triangle defect* is the element

$$\kappa_\ell := \tau(e_{rp})\,\tau(e_{qr})\,\tau(e_{pq}) \in H_*.$$

Lemma 9.3.5 (Functorial realization of triangle defect). *For every directed triangle loop ℓ,*

$$\mathrm{Hol}(\ell) = \kappa_\ell.$$

Proof. By definition of the functor Hol,

$$\mathrm{Hol}(\ell) = \mathrm{Hol}(e_{rp})\,\mathrm{Hol}(e_{qr})\,\mathrm{Hol}(e_{pq}) = \tau(e_{rp})\,\tau(e_{qr})\,\tau(e_{pq}) = \kappa_\ell.$$

□

Definition 9.3.6 (Triangle coherence quotient)**.** Let

$$Q_\triangle : \mathrm{Path}^\pm(\mathcal{N}) \twoheadrightarrow \mathrm{Path}^\pm_\triangle(\mathcal{N})$$

denote the quotient of $\mathrm{Path}^\pm(\mathcal{N})$ by the smallest groupoid congruence for which every directed triangle loop becomes the identity at its base vertex. Thus every morphism of the form

$$p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p$$

is identified with

$$\mathrm{id}_p,$$

and the quotient is closed under composition and conjugation in the groupoid sense.

Definition 9.3.7 (Based loop groups)**.** For each base vertex $p \in V$, define

$$\Omega_p(\mathcal{N}) := \mathrm{Hom}_{\mathrm{Path}^\pm(\mathcal{N})}(p,p), \qquad \Omega_p^\triangle(\mathcal{N}) := \mathrm{Hom}_{\mathrm{Path}^\pm_\triangle(\mathcal{N})}(p,p).$$

These are groups under composition.

Definition 9.3.8 (Triangle normal subgroup)**.** Fix $p \in V$. Let

$$N_{\triangle,p} \trianglelefteq \Omega_p(\mathcal{N})$$

be the normal subgroup generated by all conjugates

$$\gamma^{-1}\ell\gamma$$

where ℓ is a directed triangle loop based at some vertex $q \in V$, and

$$\gamma : p \to q$$

is any morphism in $\mathrm{Path}^\pm(\mathcal{N})$.

Theorem 9.3.9 (Loop-group quotient by triangle coherence)**.** *For each base vertex $p \in V$, the quotient functor*

$$Q_\triangle : \mathrm{Path}^\pm(\mathcal{N}) \twoheadrightarrow \mathrm{Path}^\pm_\triangle(\mathcal{N})$$

induces a surjective group homomorphism

$$(Q_\triangle)_p : \Omega_p(\mathcal{N}) \twoheadrightarrow \Omega_p^\triangle(\mathcal{N}),$$

whose kernel is exactly $N_{\triangle,p}$. Consequently there is a canonical isomorphism

$$\Omega_p^\triangle(\mathcal{N}) \cong \Omega_p(\mathcal{N})/N_{\triangle,p}.$$

Proof. Any functor between groupoids induces a homomorphism on based loop groups, so $Q_\triangle$ induces

$$(Q_\triangle)_p : \Omega_p(\mathcal{N}) \to \Omega_p^\triangle(\mathcal{N}).$$

Surjectivity follows from surjectivity of $Q_\triangle$ on morphisms. By construction of the quotient groupoid, the only new relations imposed beyond those already present in the free path groupoid are exactly the triangle coherence relations, together with all their consequences under composition and conjugation. On the based loop group at p, these consequences are precisely the normal closure of all conjugates of directed triangle loops transported to the basepoint p. Hence

$$\ker((Q_\triangle)_p) = N_{\triangle,p}.$$

The first isomorphism theorem therefore yields

$$\Omega_p^\triangle(\mathcal{N}) \cong \Omega_p(\mathcal{N})/N_{\triangle,p}.$$

□

Remark 9.3.10 (Why the quotient is necessary). Directed triangle loops are not relations in the free path groupoid. Before quotienting, they are merely loops. The passage to $\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$ is exactly the imposition of triangle coherence. The bridge proved below compares finite triangle obstruction with the *descent* of transport through this coherence quotient, not with an a priori functor already defined on the quotient.

9.4 Gauge invariance of triangle defect

Theorem 9.4.1 (Gauge invariance of triangle defect)**.** *Let ℓ be a directed triangle subsystem of a finite transport subsystem extracted from a tree-normalized connected lift. Under any change of:*

1. *base representative x_*,*
2. *tree normalization,*
3. *embedding of the subsystem into a larger finite transport system,*

the triangle defect transforms by conjugation in the stabilizer group:

$$\kappa_\ell \mapsto h\,\kappa_\ell\,h^{-1} \qquad (h \in H_*).$$

In particular,

$$\kappa_\ell = e \quad or \quad \kappa_\ell \neq e$$

is invariant.

Proof. Under any admissible change of normalization, edge labels transform by vertex relabeling:

$$\tau(e : u \to v) \mapsto h_v \, \tau(e) \, h_u^{-1},$$

for some choice of elements $h_v \in H_*$ at the vertices. Let

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big).$$

Then

$$\kappa_\ell = \tau(e_{rp}) \, \tau(e_{qr}) \, \tau(e_{pq}).$$

After relabeling,

$$\begin{aligned}\kappa_\ell &\mapsto \big(h_p \tau(e_{rp}) h_r^{-1}\big)\big(h_r \tau(e_{qr}) h_q^{-1}\big)\big(h_q \tau(e_{pq}) h_p^{-1}\big)\\ &= h_p \, \tau(e_{rp}) \, \tau(e_{qr}) \, \tau(e_{pq}) \, h_p^{-1}\\ &= h_p \, \kappa_\ell \, h_p^{-1}.\end{aligned}$$

Thus the defect transforms by conjugation. Conjugation preserves the identity element, so triviality or nontriviality is invariant. □

Remark 9.4.2 (Canonical obstruction class). By Theorem 9.4.1, a directed triangle carries a canonical conjugacy class

$$[\kappa_\ell] \in H_*/\text{conj}.$$

The intrinsic content of the obstruction is precisely the statement

$$[\kappa_\ell] = [e] \quad \text{or} \quad [\kappa_\ell] \neq [e].$$

9.5 Descent through triangle coherence

We now identify the exact obstruction to transport descent through triangle coherence.

Definition 9.5.1 (Descent through triangle coherence)**.** The transport functor

$$\text{Hol} : \text{Path}^{\pm}(\mathcal{N}) \to BH_*$$

is said to *descend through triangle coherence* if there exists a functor

$$\text{Hol}_\triangle : \text{Path}^{\pm}_\triangle(\mathcal{N}) \to BH_*$$

such that

$$\text{Hol} = \text{Hol}_\triangle \circ Q_\triangle.$$

Theorem 9.5.2 (Descent criterion)**.** *The transport functor* Hol *descends through triangle coherence if and only if*

$$\text{Hol}(\ell) = e \qquad \textit{for every directed triangle loop } \ell.$$

Equivalently, for a base vertex $p \in V$,

$$N_{\triangle,p} \subseteq \ker(\mathrm{Hol}_p),$$

where

$$\mathrm{Hol}_p : \Omega_p(\mathcal{N}) \to H_*$$

is the induced based-loop homomorphism. In particular, for a directed triangle subsystem

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big),$$

one has

$$\mathrm{Hol}(\ell) = e \quad \Longleftrightarrow \quad \kappa_\ell = e.$$

Proof. Suppose first that Hol descends through $Q_\triangle$, say

$$\mathrm{Hol} = \mathrm{Hol}_\triangle \circ Q_\triangle.$$

If ℓ is any directed triangle loop based at p, then

$$Q_\triangle(\ell) = \mathrm{id}_p\,.$$

Therefore

$$\mathrm{Hol}(\ell) = \mathrm{Hol}_\triangle(Q_\triangle(\ell)) = \mathrm{Hol}_\triangle(\mathrm{id}_p) = e.$$

Conversely, assume that

$$\mathrm{Hol}(\ell) = e \qquad \text{for every directed triangle loop } \ell.$$

Then every generator of the congruence defining $\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$ is sent by Hol to an identity morphism in BH_*. Since BH_* is a groupoid, the same holds for all consequences of these relations under composition and conjugation. Hence Hol is constant on every congruence class defining the triangle quotient. By the universal property of the quotient groupoid, there exists a unique functor

$$\mathrm{Hol}_\triangle : \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) \to BH_*$$

such that

$$\mathrm{Hol} = \mathrm{Hol}_\triangle \circ Q_\triangle.$$

This proves the first equivalence. For the based-loop formulation, the descended functor exists exactly when every element of the normal subgroup generated by triangle loops is killed by Hol_p. By Theorem 9.3.9, this is equivalent to

$$N_{\triangle,p} \subseteq \ker(\mathrm{Hol}_p).$$

Finally, for a directed triangle subsystem, Theorem 9.3.5 gives

$$\mathrm{Hol}(\ell) = \kappa_\ell.$$

Hence

$$\mathrm{Hol}(\ell) = e \quad \Longleftrightarrow \quad \kappa_\ell = e.$$

□

Lemma 9.5.3 (Basepoint independence of descent)**.** *The condition*

$$N_{\triangle,p} \subseteq \ker(\mathrm{Hol}_p)$$

is independent of the choice of base vertex $p \in V$.

Proof. Let $p, q \in V$, and let

$$\gamma : p \to q$$

be any morphism in $\mathrm{Path}^{\pm}(\mathcal{N})$. Conjugation by γ defines a group isomorphism

$$c_\gamma : \Omega_p(\mathcal{N}) \to \Omega_q(\mathcal{N}), \qquad c_\gamma(\omega) = \gamma\,\omega\,\gamma^{-1}.$$

This isomorphism sends conjugates of directed triangle loops based at p to conjugates of directed triangle loops based at q. Hence

$$c_\gamma(N_{\triangle,p}) = N_{\triangle,q}.$$

By functoriality of Hol,

$$\mathrm{Hol}_q(c_\gamma(\omega)) = \mathrm{Hol}(\gamma)\,\mathrm{Hol}_p(\omega)\,\mathrm{Hol}(\gamma)^{-1}.$$

Therefore

$$\mathrm{Hol}_p(\omega) = e \quad\Longleftrightarrow\quad \mathrm{Hol}_q(c_\gamma(\omega)) = e.$$

Thus

$$N_{\triangle,p} \subseteq \ker(\mathrm{Hol}_p) \quad\Longleftrightarrow\quad N_{\triangle,q} \subseteq \ker(\mathrm{Hol}_q).$$

So the condition is independent of the base vertex. □

Definition 9.5.4 (Quotient loop defect after descent)**.** Assume Hol descends through triangle coherence. For each base vertex $p \in V$, let

$$\delta_p^{\triangle} : \Omega_p^{\triangle}(\mathcal{N}) \to H_*$$

denote the based-loop homomorphism induced by the descended functor

$$\mathrm{Hol}_{\triangle} : \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) \to BH_*.$$

Theorem 9.5.5 (Route dependence equals loop defect on the descended quotient)**.** *Assume* Hol *descends through triangle coherence, and let*

$$\mathrm{Hol}_{\triangle} : \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) \to BH_*$$

be the descended functor. Then for a base vertex $p \in V$*, transport on*

$$\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$$

is endpoint-determined if and only if every loop defect in

$$\Omega_p^{\triangle}(\mathcal{N})$$

is trivial. By Theorem 9.5.3, this condition is independent of the choice of p.

Proof. Suppose transport on $\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$ is not endpoint-determined. Then there exist co-terminal morphisms

$$\gamma_1, \gamma_2 : p \to q$$

such that

$$\mathrm{Hol}_{\triangle}(\gamma_1) \neq \mathrm{Hol}_{\triangle}(\gamma_2).$$

Since the target BH_* is a groupoid, $\mathrm{Hol}_{\triangle}(\gamma_2)$ is invertible, and therefore

$$\mathrm{Hol}_{\triangle}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{Hol}_{\triangle}(\gamma_2)^{-1}\mathrm{Hol}_{\triangle}(\gamma_1) \neq e.$$

Hence the loop

$$\gamma_2^{-1} \circ \gamma_1 \in \Omega_p^{\triangle}(\mathcal{N})$$

has nontrivial defect. Conversely, if some loop

$$\ell \in \Omega_p^{\triangle}(\mathcal{N})$$

satisfies

$$\delta_p^{\triangle}(\ell) = \mathrm{Hol}_{\triangle}(\ell) \neq e,$$

then the co-terminal morphisms ℓ and id_p have different transport values. Thus transport is not endpoint-determined. □

9.6 Coherent vertex gauges on directed triangles

We now isolate the exact three-vertex obstruction.

Definition 9.6.1 (Directed triangle subsystem)**.** A *directed triangle subsystem* of $\mathcal{N}$ is a full three-vertex directed subgraph

$$p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p.$$

Its transport labels are

$$a := \tau(e_{pq}), \qquad b := \tau(e_{qr}), \qquad c := \tau(e_{rp}),$$

and its triangle defect is

$$\kappa_\ell = cba \in H_*.$$

Definition 9.6.2 (Coherent vertex gauge on a directed triangle)**.** Let

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big)$$

be a directed triangle subsystem. A *coherent vertex gauge* on ℓ is a choice of elements

$$u_p, u_q, u_r \in H_*$$

such that

$$\begin{aligned} u_q &= \tau(e_{pq})\, u_p, \\ u_r &= \tau(e_{qr})\, u_q, \\ u_p &= \tau(e_{rp})\, u_r. \end{aligned}$$

Lemma 9.6.3 (Pairwise coherence on a directed triangle)**.** *Every two-edge restriction of a directed triangle subsystem admits a coherent vertex gauge.*

Proof. Fix a directed triangle subsystem

$$p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p.$$

For any chosen two-edge restriction, select an arbitrary source value in H_* at the initial vertex of the first chosen edge. The transport label on that edge then determines the target value uniquely, and the transport label on the second chosen edge determines the next value uniquely. Thus each two-edge restriction admits a coherent vertex gauge. □

Proposition 9.6.4 (Global coherence on a directed triangle)**.** *Let*

$$\ell = \big(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\big)$$

be a directed triangle subsystem. Then the following are equivalent.

(i) ℓ *admits a coherent vertex gauge.*

(ii) *Its triangle defect is trivial:*

$$\kappa_\ell = \tau(e_{rp})\,\tau(e_{qr})\,\tau(e_{pq}) = e.$$

(iii) *Transport on the free path groupoid of this three-vertex subsystem descends through its triangle coherence quotient.*

Proof. (i)⇒(ii). Assume a coherent vertex gauge u_p, u_q, u_r exists. Then

$$u_q = \tau(e_{pq})u_p, \qquad u_r = \tau(e_{qr})u_q, \qquad u_p = \tau(e_{rp})u_r.$$

Substituting successively gives

$$u_p = \tau(e_{rp})\,\tau(e_{qr})\,\tau(e_{pq})\,u_p = \kappa_\ell u_p.$$

Multiplying on the right by u_p^{-1} yields

$$\kappa_\ell = e.$$

(ii)⇒(i). Assume $\kappa_\ell = e$. Choose any $u_p \in H_*$, and define

$$u_q := \tau(e_{pq})u_p, \qquad u_r := \tau(e_{qr})u_q.$$

Then

$$\tau(e_{rp})u_r = \tau(e_{rp})\,\tau(e_{qr})\,\tau(e_{pq})\,u_p = \kappa_\ell u_p = u_p.$$

Hence u_p, u_q, u_r form a coherent vertex gauge. (ii)⟺(iii). Apply Theorem 9.5.2 to the present three-vertex subsystem. By Theorem 9.3.5, the unique directed triangle loop ℓ satisfies

$$\mathrm{Hol}(\ell) = \kappa_\ell.$$

Therefore transport descends through triangle coherence if and only if

$$\mathrm{Hol}(\ell) = e,$$

which is equivalent to

$$\kappa_\ell = e.$$

□

Corollary 9.6.5 (Directed triangles are the minimal coherent obstruction)**.** *For a directed triangle subsystem, every two-edge restriction is coherently gaugeable, while the full three-edge system fails to admit a coherent vertex gauge if and only if the triangle defect is nontrivial. Equivalently, the three-edge system is the minimal obstruction to descent through triangle coherence on that subsystem.*

Proof. The pairwise statement is Theorem 9.6.3. The full three-edge statement is the contrapositive of Theorem 9.6.4. □

9.7 Single-object reformulation of the Chapter 7 triangle regime

We now identify the exact relation between the diagonal obstruction of chapter 7 and coherent vertex gauges on a finite subsystem of a tree-normalized lift.

Proposition 9.7.1 (Diagonal alignment versus coherent vertex gauge)**.** *Let $\mathcal{N} = (V, E)$ be a finite transport subsystem extracted from a tree-normalized connected lift. Then the existence of a common diagonal gauge on the corresponding finite subsystem is equivalent to the existence of a coherent vertex gauge on $\mathcal{N}$, namely an assignment*

$$u_v \in H_* \qquad (v \in V)$$

such that for every directed edge

$$e : u \to v$$

one has

$$u_v = \tau(e)\, u_u.$$

Proof. Because the lift is tree-normalized, every object representative on the finite subsystem is equal to the single base representative x_*, and every transport label lies in the stabilizer group $H_* = \mathrm{Stab}(x_*)$. A choice of vertex labels $u_v \in H_*$ therefore acts entirely within the single orbit of x_*. By the diagonal-alignment criterion of chapter 7, common diagonal gaugeability is exactly the existence of a single compatible choice of representatives across the finite subsystem. In the present single-object regime, such compatibility along an edge

$$e : u \to v$$

means precisely that the target label is obtained from the source label by the transport element $\tau(e)$:

$$u_v = \tau(e)\, u_u.$$

Thus the existence of a common diagonal gauge is equivalent to the existence of a coherent vertex gauge. □

Corollary 9.7.2 (Triangle regime in the single-object transport model). *On a directed triangle subsystem extracted from a tree-normalized connected lift, the triangle regime of chapter 7 is equivalent to pairwise coherent gaugeability together with failure of a coherent vertex gauge on the full triangle.*

Proof. By Theorem 9.7.1, common diagonal gaugeability on a finite subsystem is equivalent to coherent vertex gaugeability. Apply this to the two-edge restrictions and to the full three-edge subsystem. □

9.8 Triangle–loop identification as the rigidity bridge

We can now state the bridge theorem in exact form.

Theorem 9.8.1 (Triangle–loop identification). *Under the standing principle Standing Principle 1, let*

$$\ell = \bigl(p \xrightarrow{e_{pq}} q \xrightarrow{e_{qr}} r \xrightarrow{e_{rp}} p\bigr)$$

be a directed triangle subsystem of a finite transport subsystem extracted from a tree-normalized connected lift. Then the following are equivalent.

(1) *The corresponding three-point subsystem is in the triangle regime of chapter 7: every pairwise restriction admits a common diagonal gauge, but the full triple does not.*

(2) *The directed triangle ℓ admits coherent gauges on every two-edge restriction, but no coherent vertex gauge on the full triangle.*

(3) *The triangle defect is nontrivial:*

$$\kappa_\ell = \tau(e_{rp})\, \tau(e_{qr})\, \tau(e_{pq}) \neq e.$$

(4) *The canonical conjugacy class is nontrivial:*

$$[\kappa_\ell] \neq [e] \qquad \textit{in } H_*/\mathrm{conj}.$$

(5) *Transport on the free path groupoid of this three-vertex subsystem does not descend through its triangle coherence quotient.*

If, in addition, $\kappa_\ell = e$, so that descent exists, then the descended transport on the triangle quotient is endpoint-determined if and only if its quotient loop defects are trivial.

Proof. (1)$\Longleftrightarrow$(2). This is Theorem 9.7.2. (2)$\Longleftrightarrow$(3). By Theorem 9.6.5, every two-edge restriction is coherently gaugeable, while the full triangle fails to admit a coherent vertex gauge if and only if $\kappa_\ell \neq e$. (3)$\Longleftrightarrow$(4). By Theorem 9.4.1, κ_ℓ transforms only by conjugation under admissible changes of normalization. Hence

$$\kappa_\ell = e \quad \Longleftrightarrow \quad [\kappa_\ell] = [e],$$

and therefore

$$\kappa_\ell \neq e \quad \Longleftrightarrow \quad [\kappa_\ell] \neq [e].$$

(3)$\Longleftrightarrow$(5). By Theorem 9.6.4,

$$\kappa_\ell = e \quad \Longleftrightarrow \quad \text{transport descends through triangle coherence on the subsystem.}$$

Negating both sides yields

$$\kappa_\ell \neq e \quad \Longleftrightarrow \quad \text{transport does not descend through triangle coherence.}$$

The final statement is exactly Theorem 9.5.5, applied in the case where descent exists. □

Corollary 9.8.2 (Structural closure at the triangle boundary)**.** *On the minimal three-vertex subsystem carrying a finite diagonal obstruction, failure of transport descent through triangle coherence is exactly the first finite morphism-level witness of global transport incompatibility. Equivalently, the first finite witness of failure of transport descent through triangle coherence is triangular.*

Proof. By Theorem 9.8.1, on a directed triangle subsystem the triangle regime of chapter 7 is equivalent to failure of transport descent through triangle coherence. Since Theorem 7.5.2 identifies the triangle regime as the first strictly minimal finite witness of global incompatibility, the same subsystem is the first finite witness of failure of such descent. □

9.9 Interpretation

The bridge established in this chapter is exact at the level at which it is claimed. The finite diagonal obstruction of chapter 7 is not merely analogous to loop theory. On the minimal directed triangle subsystem, it is exactly the obstruction to descending transport through triangle coherence. Thus the first intrinsically multi-point failure of quotient semantics is already groupoidal. At the same time, the chapter makes no unsupported claim about graded depth, commutator filtrations, or quadratic carriers. Those belong to later chapters, and require their own algebraic carriers and proofs. The role of the present chapter is narrower and more rigid: it proves that the first finite diagonal obstruction and the first obstruction to transport descent coincide on the minimal triangle subsystem on which the obstruction is carried.

Remark 9.9.1 (What is not yet used). Nothing in the present chapter uses smooth realization, curvature, connections, augmentation filtrations, or quadratic carriers. The role of the chapter is exactly to identify the correct finite combinatorial obstruction and to show that, after triangle coherence is specified, this obstruction is precisely failure of transport descent on the corresponding triangle subsystem. The immediate continuation is chapter 10, where this same static obstruction is carried into observer-level irreversible descent; chapter 11 then extends that continuation before the later curvature and interface chapters chapters 12 and 13 begin from it, but none of those later chapters is used here.

9.10 Conclusion

The bridge theorem closes the discrete obstruction arc with theorem-level precision: on the minimal triangle subsystem, finite triangular incompatibility, nontrivial loop defect, and failure of transport descent are equivalent manifestations of a single obstruction class.

Accordingly, chapter 10 carries this same class into the temporal regime, where descent failure becomes the structural mechanism of irreversible dynamics; chapter 11 then continues that residue through the stitching construction before the later smooth realization chapters chapters 12 and 13.

Part IV

Dynamics, Irreversibility, and Spacetime

Chapter 10

Observer-Level Irreversible Descent and the Two-Locus Residue

10.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter sets up the internal observer-reduction framework at fixed quotient semantics, i.e. relative to items (SP4) and (SP5), and isolates the conditional regime in which observer-level irreversibility can arise. Using the static transport obstruction of chapter 9, it sets up the internal observer-reduction framework and the conditional irreversibility criterion carried next into chapter 11; only later, together with the stabilized interface-side package, do those observer-level structures feed chapter 14. Chapters 2, 3, 6, 8 and 9 established two structural facts. First, closed-system comparison semantics determines a canonical diagonal quotient

$$\pi\colon X \to \mathsf{Phys} := X/G,$$

which removes precisely the redundant degrees of freedom coming from the intrinsic diagonal symmetry. This quotient is internal to the closed system. It is not an observer coarse-graining, and by itself it creates no arrow of time. Second, any enrichment of quotient-level protocol data back to representatives in X remains exhausted by the same two previously classified loci, possibly both and with no third enrichment locus: representative choice and morphism-level transport or holonomy data. The purpose of the present chapter is to determine how this quotient architecture separates intrinsic reversible descent from the later conditional observer-induced descent regime. There are two logically distinct descent stages:

$$X \xrightarrow{\pi} \mathsf{Phys} \xrightarrow{\theta} Y.$$

(i) The intrinsic quotient

$$\pi\colon X \to \mathsf{Phys}$$

preserves reversibility: a reversible G-equivariant flow on X descends to a reversible flow on Phys.

(ii) Irreversibility may arise only after a further internal observer reduction

$$\theta\colon \mathsf{Phys} \twoheadrightarrow Y,$$

built from restricted admissible quotient-level observations; the chapter's irreversibility criterion then applies only once a proper such observer family is already forward-compatible with the descended reversible flow on Phys.

The first task of the chapter is to isolate this two-stage architecture and, once a compatible irreversible descent is fixed, derive the resulting filtration of indistinguishability relations on Phys. From that filtration follow two canonical consequences: a monotone entropy functional and an extended pseudo-ultrametric geometry. The second task is structural. We prove that irreversibility at the observer level introduces no new enrichment primitive. Even after the reduction θ, every compositional lift of quotient-level data back to representatives in X remains exhausted by the same two loci identified in Theorems 6.6.1 and 6.6.2: representative choice and morphism-level transport or holonomy data, possibly both, with no third enrichment locus. The central structural claim is therefore the following: once a G-equivariant reversible flow on X and its descended reversible flow on Phys are fixed, nontrivial transport obstruction forces irreversible observer descent whenever a proper internal observer family built from restricted quotient-level observations is already forward-compatible with that descended flow. **Dependence on labeled results.** This chapter uses:

- the intrinsic quotient semantics $X \to \mathsf{Phys}$ established in the quotient chapters;
- representative lift classification and the two-locus analysis from Chapters 6 and 8;
- the triangle–loop bridge and nontrivial transport obstruction from Chapter 9.

Use in subsequent chapters. This chapter provides:

- the rigid separation between intrinsic reversible quotient semantics and the later observer-level descent on Phys where irreversible behavior may arise;
- the internal observer-reduction framework on Phys, together with the conditional irreversibility criterion for proper observer families built from restricted quotient-level protocols once forward compatibility is fixed;
- the canonical filtration of observer indistinguishability on Phys attached to a compatible irreversible descent;
- the entropy and extended pseudo-ultrametric structures induced by that filtration;
- the theorem that even in the irreversible regime the enrichment residue above quotient semantics remains exhausted by the same two loci: representative choice and morphism-level transport or holonomy data, possibly both, with no third locus.

10.2 Reversible flows and forward factors

We begin with the abstract distinction between reversible dynamics and forward-only reduced dynamics.

Definition 10.2.1 (Reversible flow). A *reversible flow* on a set X is a family

$$(\Phi_t)_{t\in\mathbb{R}}$$

of bijections $\Phi_t\colon X \to X$ satisfying

$$\Phi_0 = \mathrm{id}_X, \qquad \Phi_{t+s} = \Phi_t \circ \Phi_s \qquad \text{for all } s,t \in \mathbb{R}.$$

Definition 10.2.2 (Forward factor map). Let $(\Phi_t)_{t\in\mathbb{R}}$ be a reversible flow on X. A surjection

$$\rho\colon X \twoheadrightarrow Y$$

is a *forward factor map* if for every $t \geq 0$ there exists a map

$$\Psi_t\colon Y \to Y$$

such that

$$\rho \circ \Phi_t = \Psi_t \circ \rho.$$

Lemma 10.2.3 (Forward factor semigroup). *If $\rho\colon X \twoheadrightarrow Y$ is a forward factor map, then the family $(\Psi_t)_{t\geq 0}$ satisfies*

$$\Psi_0 = \mathrm{id}_Y, \qquad \Psi_{t+s} = \Psi_t \circ \Psi_s \qquad \textit{for all } s,t \geq 0.$$

Proof. Since

$$\rho \circ \Phi_0 = \rho,$$

surjectivity of ρ implies

$$\Psi_0 = \mathrm{id}_Y\,.$$

Let $s,t \geq 0$. Then

$$\Psi_t \circ \Psi_s \circ \rho = \Psi_t \circ \rho \circ \Phi_s = \rho \circ \Phi_t \circ \Phi_s = \rho \circ \Phi_{t+s} = \Psi_{t+s} \circ \rho.$$

Again by surjectivity of ρ,

$$\Psi_{t+s} = \Psi_t \circ \Psi_s.$$

□

Definition 10.2.4 (Information loss). A forward factor map ρ *loses information* if there exist distinct points $x_1, x_2 \in X$ and a time $t_0 > 0$ such that

$$\rho(x_1) = \rho(x_2)$$

but

$$\rho(\Phi_{-t_0}(x_1)) \neq \rho(\Phi_{-t_0}(x_2)).$$

Thus ρ identifies states which remain distinguishable when the underlying reversible dynamics is run backward.

Lemma 10.2.5 (Information loss obstructs two-sided descent)**.** *If ρ loses information, then there does not exist a family*

$$(\widetilde{\Psi}_t)_{t\in\mathbb{R}}$$

extending $(\Psi_t)_{t\geq 0}$ and satisfying

$$\rho \circ \Phi_t = \widetilde{\Psi}_t \circ \rho \qquad \textit{for all } t \in \mathbb{R}.$$

Proof. Assume for contradiction that such an extension exists. Choose x_1, x_2, t_0 witnessing information loss. Evaluating the intertwining identity at time $-t_0$ gives

$$\widetilde{\Psi}_{-t_0}(\rho(x_1)) = \rho(\Phi_{-t_0}(x_1)), \qquad \widetilde{\Psi}_{-t_0}(\rho(x_2)) = \rho(\Phi_{-t_0}(x_2)).$$

Since $\rho(x_1) = \rho(x_2)$, the two left-hand sides are equal, whereas the two right-hand sides are distinct by hypothesis. This is impossible. □

Remark 10.2.6 (Irreversibility mechanism)*.* Noninjectivity of ρ alone does not force irreversibility. The obstruction is the backward fiber separation of Theorem 10.2.4. Irreversibility appears exactly when states that are identified at the reduced level fail to remain identified under backward evolution in the unreduced reversible system.

10.3 The intrinsic quotient preserves reversibility

We now apply the abstract discussion to the closed-system quotient architecture.

Definition 10.3.1 (G-equivariant flow)**.** A reversible flow $(\Phi_t)_{t\in\mathbb{R}}$ on X is *G-equivariant* if

$$\Phi_t(g \cdot x) = g \cdot \Phi_t(x) \qquad \text{for all } g \in G,\ x \in X,\ t \in \mathbb{R}.$$

Proposition 10.3.2 (Time descends through the intrinsic quotient)**.** *Suppose $(\Phi_t)_{t\in\mathbb{R}}$ is a G-equivariant reversible flow on X. Then there exists a unique reversible flow*

$$(\widehat{\Psi}_t)_{t\in\mathbb{R}}$$

on Phys *such that*

$$\pi \circ \Phi_t = \widehat{\Psi}_t \circ \pi \qquad \textit{for all } t \in \mathbb{R}.$$

Proof. Define

$$\widehat{\Psi}_t([x]) := [\Phi_t(x)].$$

To prove well-definedness, suppose $[x] = [y]$. Then $y = g \cdot x$ for some $g \in G$. By equivariance,

$$\Phi_t(y) = \Phi_t(g \cdot x) = g \cdot \Phi_t(x),$$

so $[\Phi_t(y)] = [\Phi_t(x)]$. Thus $\widehat{\Psi}_t$ is well defined. For every $x \in X$,

$$(\widehat{\Psi}_t \circ \pi)(x) = \widehat{\Psi}_t([x]) = [\Phi_t(x)] = (\pi \circ \Phi_t)(x),$$

so the intertwining identity holds. The flow laws follow directly from those of (Φ_t):

$$\widehat{\Psi}_0([x]) = [x],$$

and

$$\widehat{\Psi}_{t+s}([x]) = [\Phi_{t+s}(x)] = [\Phi_t(\Phi_s(x))] = \widehat{\Psi}_t(\widehat{\Psi}_s([x])).$$

Since each Φ_t is bijective, each $\widehat{\Psi}_t$ is bijective with inverse $\widehat{\Psi}_{-t}$. Hence $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ is a reversible flow on Phys. Uniqueness follows from surjectivity of π. □

Corollary 10.3.3 (No arrow from the intrinsic quotient)**.** *If*

$$\pi(x_1) = \pi(x_2),$$

then for every $t \geq 0$ one has

$$\pi(\Phi_{-t}(x_1)) = \pi(\Phi_{-t}(x_2)).$$

Proof. By Theorem 10.3.2, the descended reversible flow $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ exists on Phys. Therefore

$$\pi(\Phi_{-t}(x_1)) = \widehat{\Psi}_{-t}(\pi(x_1)) = \widehat{\Psi}_{-t}(\pi(x_2)) = \pi(\Phi_{-t}(x_2)).$$

□

Thus the intrinsic diagonal quotient preserves reversibility. Any arrow of time must therefore arise only after a later internal observer reduction

$$\theta \colon \mathsf{Phys} \twoheadrightarrow Y.$$

10.4 Internal observer reductions

The second descent stage must be internal to the closed system. Accordingly, we now set up the internal observer-reduction framework using restricted quotient-level observation protocols.

Definition 10.4.1 (Admissible quotient-level observable)**.** An *admissible quotient-level observable* is a map

$$\omega \colon \mathsf{Phys} \to V$$

which arises from an admissible quotient-level protocol functor and is invariant under the transport equivalences already determined at the quotient level.

Definition 10.4.2 (Observable family)**.** A *quotient-level observable family* is a collection

$$\mathfrak{O} = \{\omega_a \colon \mathsf{Phys} \to V_a\}_{a \in A}$$

of admissible quotient-level observables.

Definition 10.4.3 (Observer indistinguishability)**.** Given a quotient-level observable family $\mathfrak{O}$, define a relation $\sim_{\mathfrak{O}}$ on Phys by

$$p \sim_{\mathfrak{O}} q \iff \omega_a(p) = \omega_a(q) \qquad \text{for all } a \in A.$$

Lemma 10.4.4 (Observer indistinguishability is an equivalence relation)**.** *For every quotient-level observable family $\mathfrak{O}$, the relation $\sim_{\mathfrak{O}}$ is an equivalence relation on* Phys*.*

Proof. Reflexivity and symmetry are immediate. For transitivity, suppose

$$p \sim_{\mathfrak{O}} q \qquad \text{and} \qquad q \sim_{\mathfrak{O}} r.$$

Then for every $a \in A$,

$$\omega_a(p) = \omega_a(q) = \omega_a(r).$$

Hence $p \sim_{\mathfrak{O}} r$. □

Definition 10.4.5 (Internal observer reduction)**.** Let $\mathfrak{O}$ be a quotient-level observable family. Define

$$Y_{\mathfrak{O}} := \mathsf{Phys}/{\sim_{\mathfrak{O}}},$$

and let

$$\theta_{\mathfrak{O}} \colon \mathsf{Phys} \twoheadrightarrow Y_{\mathfrak{O}}$$

be the canonical quotient map

$$\theta_{\mathfrak{O}}(p) := [p]_{\sim_{\mathfrak{O}}}.$$

This map is called the *internal observer reduction* induced by $\mathfrak{O}$.

Remark 10.4.6 (Internality). The map $\theta_{\mathfrak{O}}$ is not an externally imposed coarse-graining. It is determined entirely by a restricted family of admissible quotient-level observations already internal to the closed system.

Proposition 10.4.7 (Universal property of observer reduction)**.** *For every $\omega_a \in \mathfrak{O}$, there exists a unique map*

$$\overline{\omega}_a \colon Y_{\mathfrak{O}} \to V_a$$

such that

$$\omega_a = \overline{\omega}_a \circ \theta_{\mathfrak{O}}.$$

Moreover, $\theta_{\mathfrak{O}}$ is initial among surjections with this property.

Proof. If $p \sim_{\mathfrak{O}} q$, then $\omega_a(p) = \omega_a(q)$, so ω_a is constant on $\sim_{\mathfrak{O}}$-classes. Therefore

$$\overline{\omega}_a([p]_{\sim_{\mathfrak{O}}}) := \omega_a(p)$$

is well defined. Uniqueness follows from surjectivity of $\theta_{\mathfrak{O}}$. Now suppose

$$\eta\colon \mathsf{Phys} \twoheadrightarrow Z$$

is another surjection through which every ω_a factors. If $\eta(p) = \eta(q)$, then $\omega_a(p) = \omega_a(q)$ for every a, hence $p \sim_{\mathfrak{O}} q$. Therefore η refines $\sim_{\mathfrak{O}}$, and there exists a unique map

$$u\colon Z \to Y_{\mathfrak{O}}$$

with

$$\theta_{\mathfrak{O}} = u \circ \eta.$$

Thus $\theta_{\mathfrak{O}}$ is initial among such surjections. □

Definition 10.4.8 (Proper observer family)**.** A quotient-level observable family $\mathfrak{O}$ is *proper* if there exist distinct points $p_1, p_2 \in \mathsf{Phys}$ such that

$$p_1 \sim_{\mathfrak{O}} p_2.$$

Equivalently, the induced observer reduction $\theta_{\mathfrak{O}}$ is noninjective.

Definition 10.4.9 (Forward-compatible observer family)**.** Let $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ be a reversible flow on Phys. A quotient-level observable family $\mathfrak{O}$ is *forward-compatible* with $(\widehat{\Psi}_t)$ if for every $t \geq 0$ there exists a map

$$\Psi_t^{\mathfrak{O}}\colon Y_{\mathfrak{O}} \to Y_{\mathfrak{O}}$$

such that

$$\theta_{\mathfrak{O}} \circ \widehat{\Psi}_t = \Psi_t^{\mathfrak{O}} \circ \theta_{\mathfrak{O}}.$$

Definition 10.4.10 (Backward-stable observer family)**.** Let $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ be a reversible flow on Phys, and let $\mathfrak{O}$ be a quotient-level observable family that is forward-compatible with $(\widehat{\Psi}_t)$. We say that $\mathfrak{O}$ is *backward-stable* if the intertwining identity extends to all $t \in \mathbb{R}$, i.e. if there exists a family

$$(\widetilde{\Psi}_t^{\mathfrak{O}})_{t\in\mathbb{R}}$$

on $Y_{\mathfrak{O}}$ such that

$$\theta_{\mathfrak{O}} \circ \widehat{\Psi}_t = \widetilde{\Psi}_t^{\mathfrak{O}} \circ \theta_{\mathfrak{O}} \qquad \text{for all } t \in \mathbb{R}.$$

10.5 Transport obstruction and fixed-flow observer irreversibility

We now identify the exact point at which, relative to fixed reversible flow data, nontrivial transport obstruction forces observer-level irreversibility once the associated restricted observer family is forward-compatible.

Definition 10.5.1 (Observer-level information loss on Phys)**.** Let $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ be a reversible flow on Phys, and let $\mathfrak{O}$ be a quotient-level observable family that is forward-compatible with $(\widehat{\Psi}_t)$. We say that $\mathfrak{O}$ *loses information* if there exist distinct points $p_1, p_2 \in \mathsf{Phys}$ and a time $t_0 > 0$ such that

$$\theta_{\mathfrak{O}}(p_1) = \theta_{\mathfrak{O}}(p_2)$$

but

$$\theta_{\mathfrak{O}}(\widehat{\Psi}_{-t_0}(p_1)) \neq \theta_{\mathfrak{O}}(\widehat{\Psi}_{-t_0}(p_2)).$$

Lemma 10.5.2 (Lifted and quotient-level information loss agree)**.** *Let (Φ_t) be a G-equivariant reversible flow on X, let $(\widehat{\Psi}_t)$ be the descended reversible flow on* Phys*, and let $\mathfrak{O}$ be a quotient-level observable family on* Phys *that is forward-compatible with $(\widehat{\Psi}_t)$. Set*

$$\theta := \theta_{\mathfrak{O}}, \qquad \rho := \theta \circ \pi.$$

Then ρ loses information in the sense of Theorem 10.2.4 if and only if $\mathfrak{O}$ loses information in the sense of Theorem 10.5.1.

Proof. Assume first that ρ loses information. Then there exist $x_1, x_2 \in X$ and $t_0 > 0$ such that

$$\rho(x_1) = \rho(x_2)$$

but

$$\rho(\Phi_{-t_0}(x_1)) \neq \rho(\Phi_{-t_0}(x_2)).$$

Set

$$p_i := \pi(x_i) \in \mathsf{Phys} \qquad (i = 1, 2).$$

Then

$$\theta(p_1) = \rho(x_1) = \rho(x_2) = \theta(p_2).$$

Moreover, by Theorem 10.3.2,

$$\pi(\Phi_{-t_0}(x_i)) = \widehat{\Psi}_{-t_0}(p_i).$$

Hence

$$\theta(\widehat{\Psi}_{-t_0}(p_1)) = \rho(\Phi_{-t_0}(x_1)) \neq \rho(\Phi_{-t_0}(x_2)) = \theta(\widehat{\Psi}_{-t_0}(p_2)).$$

Thus $\mathfrak{O}$ loses information on Phys. Conversely, assume $\mathfrak{O}$ loses information on Phys. Then there exist distinct points $p_1, p_2 \in \mathsf{Phys}$ and $t_0 > 0$ such that

$$\theta(p_1) = \theta(p_2)$$

but

$$\theta(\widehat{\Psi}_{-t_0}(p_1)) \neq \theta(\widehat{\Psi}_{-t_0}(p_2)).$$

Choose representatives $x_i \in X$ with

$$\pi(x_i) = p_i \qquad (i = 1, 2).$$

Then

$$\rho(x_1) = \theta(\pi(x_1)) = \theta(p_1) = \theta(p_2) = \theta(\pi(x_2)) = \rho(x_2),$$

while again by Theorem 10.3.2,

$$\rho(\Phi_{-t_0}(x_i)) = \theta(\pi(\Phi_{-t_0}(x_i))) = \theta(\widehat{\Psi}_{-t_0}(p_i)).$$

Hence

$$\rho(\Phi_{-t_0}(x_1)) \neq \rho(\Phi_{-t_0}(x_2)).$$

Thus ρ loses information. □

Definition 10.5.3 (Compatible irreversible descent)**.** A *compatible irreversible descent* consists of:

(1) a G-equivariant reversible flow

$$(\Phi_t)_{t\in\mathbb{R}}$$

on X;

(2) a proper quotient-level observable family $\mathfrak{O}$ on Phys, forward-compatible with the descended reversible flow $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ on Phys determined by item (1) via Theorem 10.3.2;

(3) the induced internal observer reduction

$$\theta := \theta_{\mathfrak{O}} \colon \mathsf{Phys} \twoheadrightarrow Y_{\mathfrak{O}};$$

(4) information loss for $\mathfrak{O}$, equivalently for $\rho := \theta \circ \pi$, in the sense of Theorem 10.5.2.

Lemma 10.5.4 (Factor identity on Phys)**.** *Let*

$$\rho = \theta \circ \pi \colon X \twoheadrightarrow Y$$

be a compatible irreversible descent, let $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ *be the descended reversible flow on* Phys*, and let* $(\Psi_t)_{t\geq 0}$ *be the induced forward semigroup on* Y*. Then for every* $t \geq 0$*,*

$$\theta \circ \widehat{\Psi}_t = \Psi_t \circ \theta.$$

Proof. For $x \in X$ and $t \geq 0$,

$$(\theta \circ \widehat{\Psi}_t \circ \pi)(x) = (\theta \circ \pi \circ \Phi_t)(x) = (\rho \circ \Phi_t)(x) = (\Psi_t \circ \rho)(x) = (\Psi_t \circ \theta \circ \pi)(x).$$

Since π is surjective,

$$\theta \circ \widehat{\Psi}_t = \Psi_t \circ \theta.$$

□

We now state the supported conditional forcing theorem.

Theorem 10.5.5 (Conditional fixed-flow observer irreversibility)**.** *Under the standing principle Standing Principle 1, assume the closed comparison system is nontrivial in the sense that the transport obstruction of Chapter 9 is nonvanishing on some finite transport subsystem. Let $(\Phi_t)_{t\in\mathbb{R}}$ be a G-equivariant reversible flow on X, and let $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ be the descended reversible flow on* Phys*. Fix a minimal triangle subsystem on which transport fails to descend through triangle coherence, and let $\mathfrak{O}$ be the proper admissible quotient-level observable family on* Phys *whose values depend only on forward-coherent restrictions of that subsystem. Assume that $\mathfrak{O}$ is forward-compatible with $(\widehat{\Psi}_t)$. Then:*

(i) *$\mathfrak{O}$ is not backward-stable with respect to $(\widehat{\Psi}_t)$;*

(ii) *the induced internal observer reduction*

$$\theta_{\mathfrak{O}} \colon \mathsf{Phys} \twoheadrightarrow Y_{\mathfrak{O}}$$

loses information in the sense of Theorem 10.5.1;

(iii) *the composite*

$$\rho := \theta_{\mathfrak{O}} \circ \pi$$

loses information on X, hence defines a compatible irreversible descent.

Proof. By Theorem 9.8.1, nontrivial transport obstruction means that there exists a finite transport subsystem carrying a minimal triangle regime, equivalently a directed triangle subsystem on which transport fails to descend through triangle coherence. Fix such a subsystem, and let $\mathfrak{O}$ be the associated admissible quotient-level observable family from the statement. This family is internal, because it is defined entirely in terms of admissible quotient-level protocols, and it is proper because the full triangle obstruction is invisible on every two-edge restriction while distinct global states remain. Let $\theta_{\mathfrak{O}} \colon \mathsf{Phys} \twoheadrightarrow Y_{\mathfrak{O}}$ be the induced observer reduction. By assumption, $\mathfrak{O}$ is forward-compatible with the fixed descended reversible flow $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ on Phys. We claim that $\mathfrak{O}$ is not backward-stable. Suppose instead that it were backward-stable with respect to $(\widehat{\Psi}_t)$. Then there would exist a family

$$(\widetilde{\Psi}_t^{\mathfrak{O}})_{t\in\mathbb{R}}$$

on $Y_{\mathfrak{O}}$ such that

$$\theta_{\mathfrak{O}} \circ \widehat{\Psi}_t = \widetilde{\Psi}_t^{\mathfrak{O}} \circ \theta_{\mathfrak{O}} \qquad \text{for all } t \in \mathbb{R}.$$

On the chosen minimal obstructed subsystem, the family $\mathfrak{O}$ records only the forward-coherent restricted reports. A two-sided intertwining through $\theta_{\mathfrak{O}}$ would therefore make those restricted reports stable under full transport and force the missing global coherence to descend through the observer quotient. By the triangle–loop identification of Theorem 9.8.1, that would give a two-sided transport descent across the transport-obstructed subsystem, contradicting the assumed obstruction. Hence $\mathfrak{O}$ is not backward-stable. If, for every $t > 0$ and every $p_1, p_2 \in \mathsf{Phys}$ with $\theta_{\mathfrak{O}}(p_1) = \theta_{\mathfrak{O}}(p_2)$, one also had

$$\theta_{\mathfrak{O}}(\widehat{\Psi}_{-t}(p_1)) = \theta_{\mathfrak{O}}(\widehat{\Psi}_{-t}(p_2)),$$

then each backward iterate $\widehat{\Psi}_{-t}$ would descend through the surjection $\theta_{\mathfrak{O}}$ to a map on $Y_{\mathfrak{O}}$. Together with forward compatibility for $t \geq 0$, this would make $\mathfrak{O}$ backward-stable, a contradiction. Therefore there exist distinct points $p_1, p_2 \in \mathsf{Phys}$ and a time $t_0 > 0$ such that

$$\theta_{\mathfrak{O}}(p_1) = \theta_{\mathfrak{O}}(p_2)$$

but

$$\theta_{\mathfrak{O}}(\widehat{\Psi}_{-t_0}(p_1)) \neq \theta_{\mathfrak{O}}(\widehat{\Psi}_{-t_0}(p_2)).$$

That is exactly observer-level information loss in the sense of Theorem 10.5.1. By Theorem 10.5.2, the composite

$$\rho := \theta_{\mathfrak{O}} \circ \pi$$

loses information on X as well. Since $\mathfrak{O}$ is proper and forward-compatible with the fixed descended flow, ρ is therefore a compatible irreversible descent. □

Corollary 10.5.6 (Conditional irreversibility for a fixed flow)**.** *Assume the hypotheses of Theorem 10.5.5. Then, for the fixed reversible flow data, the associated internal observer reduction $\theta_{\mathfrak{O}}$ is forward-compatible but not backward-stable and therefore exhibits observer-level irreversibility.*

Proof. Fix the G-equivariant reversible flow (Φ_t) on X, the descended reversible flow $(\widehat{\Psi}_t)$ on Phys, and the associated observer family $\mathfrak{O}$ as in Theorem 10.5.5. The theorem gives non-backward-stability and observer-level information loss for $\theta_{\mathfrak{O}}$, which is exactly the claimed irreversibility. □

Remark 10.5.7 (Internal source of the arrow). The observer reduction θ is now fully internalized: it is induced by a proper restricted family of admissible quotient-level observations. Irreversibility is therefore not postulated by an external coarse-graining. For the fixed reversible flow data of Theorem 10.5.5, it appears precisely when internal observer reduction is forward-compatible with the descended flow but not backward-stable.

10.6 Time-indexed indistinguishability on Phys

Fix a compatible irreversible descent. Thus a G-equivariant reversible flow $(\Phi_t)_{t\in\mathbb{R}}$ on X is given together with its descended reversible flow $(\widehat{\Psi}_t)_{t\in\mathbb{R}}$ on Phys, and an internal observer reduction

$$X \xrightarrow{\pi} \mathsf{Phys} \xrightarrow{\theta} Y, \qquad \rho = \theta \circ \pi.$$

Definition 10.6.1 (Kernel filtration on Phys)**.** For each $t \geq 0$, define an equivalence relation

$$\mathcal{F}_t := \ker(\theta \circ \widehat{\Psi}_t) \subseteq \mathsf{Phys} \times \mathsf{Phys}$$

by

$$(p_1, p_2) \in \mathcal{F}_t \iff \theta(\widehat{\Psi}_t(p_1)) = \theta(\widehat{\Psi}_t(p_2)).$$

Lemma 10.6.2 (Equivalence)**.** *Each $\mathcal{F}_t$ is an equivalence relation on* Phys*.*

Proof. Fix $t \geq 0$. Since $\mathcal{F}_t$ is the kernel relation of the map

$$f_t := \theta \circ \widehat{\Psi}_t \colon \mathsf{Phys} \to Y,$$

the relation

$$(p, q) \in \mathcal{F}_t \iff f_t(p) = f_t(q)$$

is reflexive, symmetric, and transitive. □

Lemma 10.6.3 (Monotone coarsening)**.** *If $0 \leq s \leq t$, then*

$$\mathcal{F}_s \subseteq \mathcal{F}_t.$$

Proof. Let $(p_1, p_2) \in \mathcal{F}_s$. By definition,

$$\theta(\widehat{\Psi}_s(p_1)) = \theta(\widehat{\Psi}_s(p_2)).$$

Write

$$t = s + (t - s).$$

Applying Ψ_{t-s} to both sides gives

$$\Psi_{t-s}\big(\theta(\widehat{\Psi}_s(p_1))\big) = \Psi_{t-s}\big(\theta(\widehat{\Psi}_s(p_2))\big).$$

Using Theorem 10.5.4,

$$\Psi_{t-s} \circ \theta = \theta \circ \widehat{\Psi}_{t-s},$$

and then the flow law on Phys, we obtain

$$\theta(\widehat{\Psi}_t(p_1)) = \theta(\widehat{\Psi}_t(p_2)).$$

Hence $(p_1, p_2) \in \mathcal{F}_t$, so

$$\mathcal{F}_s \subseteq \mathcal{F}_t.$$

□

Lemma 10.6.4 (Coarsening map between quotient partitions)**.** *If $0 \leq s \leq t$, then the identity map on* Phys *induces a canonical surjection*

$$q_{s,t} \colon \mathsf{Phys}/\mathcal{F}_s \twoheadrightarrow \mathsf{Phys}/\mathcal{F}_t, \qquad [p]_{\mathcal{F}_s} \longmapsto [p]_{\mathcal{F}_t}.$$

Proof. Well-definedness follows from Theorem 10.6.3: if

$$[p]_{\mathcal{F}_s} = [p']_{\mathcal{F}_s},$$

then

$$(p, p') \in \mathcal{F}_s \subseteq \mathcal{F}_t,$$

hence

$$[p]_{\mathcal{F}_t} = [p']_{\mathcal{F}_t}.$$

Surjectivity is immediate. □

Remark 10.6.5 (Arrow of time). The filtration

$$\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \qquad (0 \leq s \leq t)$$

grows monotonically. Thus indistinguishable states can merge but never split. The arrow of time is therefore located entirely in the observer reduction θ, not in the intrinsic quotient π.

10.7 Entropy and extended pseudo-ultrametric

Definition 10.7.1 (Observer entropy)**.** Assume that

$$|\mathsf{Phys}/\mathcal{F}_t| < \infty$$

for the times under consideration. Define

$$S_\theta(t) := -\log |\mathsf{Phys}/\mathcal{F}_t|.$$

Proposition 10.7.2 (Entropy monotonicity)**.** *Assume that* $|\mathsf{Phys}/\mathcal{F}_t| < \infty$ *for the times under consideration. Then* $S_\theta(t)$ *is nondecreasing in* t*.*

Proof. If $0 \leq s \leq t$, then by Theorem 10.6.4 there exists a surjection

$$q_{s,t} \colon \mathsf{Phys}/\mathcal{F}_s \twoheadrightarrow \mathsf{Phys}/\mathcal{F}_t.$$

Hence

$$|\mathsf{Phys}/\mathcal{F}_t| \leq |\mathsf{Phys}/\mathcal{F}_s|.$$

Applying the decreasing function $-\log$ yields

$$S_\theta(s) \leq S_\theta(t).$$

□

Remark 10.7.3 (Observer dependence). The entropy functional depends on the reduction θ. Different internal observer families induce different filtrations

$$(\mathcal{F}_t)_{t\geq 0}$$

and therefore different entropy functions. What is intrinsic is not the numerical value of $S_\theta(t)$, but its monotone direction, which follows purely from kernel growth.

Definition 10.7.4 (Observer extended pseudo-ultrametric)**.** Define a function

$$d_\theta \colon \mathsf{Phys} \times \mathsf{Phys} \to [0, \infty]$$

by

$$d_\theta(p, q) := \inf\{t \geq 0 : (p, q) \in \mathcal{F}_t\}.$$

If the displayed set is empty, we use the convention $d_\theta(p, q) = +\infty$.

Proposition 10.7.5 (Extended pseudo-ultrametric structure)**.** *The function d_θ is an extended pseudo-ultrametric on* Phys*. That is:*

(i)
$$d_\theta(p, p) = 0;$$

(ii)
$$d_\theta(p, q) = d_\theta(q, p);$$

(iii)
$$d_\theta(p, r) \leq \max\{d_\theta(p, q), d_\theta(q, r)\}.$$

Proof. Since

$$(p, p) \in \mathcal{F}_0,$$

one has

$$d_\theta(p, p) = 0.$$

Each relation $\mathcal{F}_t$ is symmetric, hence so is d_θ. For the strong triangle inequality, set

$$a := d_\theta(p, q), \qquad b := d_\theta(q, r), \qquad m := \max\{a, b\}.$$

If $m = +\infty$, then

$$d_\theta(p, r) \leq \max\{d_\theta(p, q), d_\theta(q, r)\}$$

is immediate. So assume $m < +\infty$. Let $\varepsilon > 0$. By definition of infimum, there exist

$$t_1 \leq a + \varepsilon, \qquad t_2 \leq b + \varepsilon$$

such that

$$(p, q) \in \mathcal{F}_{t_1}, \qquad (q, r) \in \mathcal{F}_{t_2}.$$

Set

$$t := \max\{t_1, t_2\} \le m + \varepsilon.$$

By Theorem 10.6.3,

$$(p, q) \in \mathcal{F}_t, \qquad (q, r) \in \mathcal{F}_t.$$

Since $\mathcal{F}_t$ is an equivalence relation, it is transitive, so

$$(p, r) \in \mathcal{F}_t.$$

Hence

$$d_\theta(p, r) \le t \le m + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ yields

$$d_\theta(p, r) \le \max\{d_\theta(p, q), d_\theta(q, r)\}.$$

□

Remark 10.7.6 (Coarsening time as geometry)*.* The extended pseudo-ultrametric d_θ packages the filtration $(\mathcal{F}_t)_{t\ge0}$ as a geometry on Phys: states are close when they become θ-indistinguishable early. Pairs that never become θ-indistinguishable simply remain at infinite distance. This geometry is observer-dependent, but the strong triangle inequality is purely filtration-theoretic.

10.8 The two-locus residue

We now return to the enrichment question. The point is that observer reduction changes distinguishability at the level of reduced dynamics, but does not change the internal classification of compositional lifts back to representatives.

Theorem 10.8.1 (Irreversible descent preserves the two-locus residue)**.** *Under the standing principle Standing Principle 1, let*

$$\rho = \theta \circ \pi \colon X \twoheadrightarrow Y$$

be a compatible irreversible descent. Then:

(1) *the reduced forward dynamics on Y is irreversible in the sense that it admits no two-sided extension intertwining ρ;*

(2) *quotient semantics remains determined by the intrinsic quotient*

$$\pi \colon X \to \mathsf{Phys};$$

(3) *every compositional lift of quotient-level protocol data to representatives in X remains exhausted by the same two previously classified loci, possibly both and with no third locus:*

(a) *representative sections;*

(b) *morphism-level transport/holonomy data.*

Proof. Item (1) is exactly Theorem 10.2.5: information loss precludes any two-sided extension of the reduced forward semigroup intertwining ρ. For item (2), the quotient semantics of the closed system is unchanged by the observer reduction. The intrinsic physical state space is $\mathsf{Phys} = X/G$, obtained from the canonical diagonal quotient. The map

$$\theta \colon \mathsf{Phys} \twoheadrightarrow Y$$

is an additional internal observer reduction imposed *after* that quotient. It may erase distinctions between physically distinct states, but it does not alter the intrinsic quotient semantics itself. For item (3), any compositional enrichment of quotient-level protocol data back to representatives in X is a representative-lift problem through the action groupoid. By the lift-classification theorem of Chapter 6 and the descent-obstruction classification of Chapter 8, such a lift is exhausted by the same two kinds of additional data: object-level representative choice and morphism-level transport cocycle or holonomy data. Since the observer reduction θ acts only after passage to Phys, it creates no new action-groupoid lift data above quotient semantics. Hence no third enrichment locus appears. □

Remark 10.8.2 (Structural summary)*.* The logical architecture is therefore rigid:

$$X \xrightarrow{\pi} \mathsf{Phys} \xrightarrow{\theta} Y.$$

The first map is intrinsic and reversible; it defines the closed-system quotient semantics. The second map is internal but observer-dependent; it may be irreversible and induces the monotone indistinguishability filtration on Phys. Yet even in the irreversible regime, representative choice and morphism-level transport or holonomy data remain the only enrichment loci above quotient semantics, possibly both, with no third locus, by Theorems 6.6.1 and 6.6.2.

10.9 Interpretation

The chapter establishes four points which should be kept sharply separate. First, the canonical quotient

$$\pi \colon X \to \mathsf{Phys}$$

is not an irreversible coarse-graining. It removes only the intrinsic diagonal redundancy determined by closed comparison semantics. Because G-equivariant reversible flows descend through π as reversible flows, no arrow of time originates there. Second, observer reduction is internal to the closed system. It is not an externally imposed surjection, but the quotient induced by a proper restricted family of admissible quotient-level observations. Thus the second descent stage

$$\theta \colon \mathsf{Phys} \twoheadrightarrow Y$$

arises from observer limitation already encoded inside the comparison stack. Third, once a reversible flow on X and its descended reversible flow on Phys are fixed, irreversibility appears only when such an internal observer reduction is forward-compatible with that descended flow but not backward-stable. The exact mechanism is backward fiber separation. This induces the monotone kernel filtration

$$(\mathcal{F}_t)_{t\geq 0}$$

on Phys, from which observer entropy and the extended pseudo-ultrametric geometry follow canonically. Fourth, the enrichment residue is unaffected by this observer-level irreversibility. Even when reduced dynamics becomes irreversible on Y, the only compositional ways to lift quotient-level data back to X remain the two loci of Theorems 6.6.1 and 6.6.2: representative choice and morphism-level transport or holonomy data, possibly both, with no third enrichment locus. In particular, the chapter introduces no new primitive for irreversible behavior. Irreversibility is carried by internal observer-induced failure of descent, not by an enlargement of the intrinsic comparison stack.

Remark 10.9.1 (What is not yet used). Nothing in the present chapter uses smooth realization, connection theory, curvature, graded commutator structure, or quadratic carriers. The role of the chapter is more basic: for fixed reversible flow data, it isolates the exact observer-level locus at which irreversibility can appear once a proper internal observer family is forward-compatible, and it proves that representative choice and morphism-level transport or holonomy data remain the only enrichment loci above quotient semantics, possibly both, with no third locus. Chapter 11 is the immediate next continuation, and chapters 14 and 17 use the stabilized arena later, but none of that is used here.

10.10 Conclusion

The chapter isolates irreversibility at its exact structural location: not in the intrinsic quotient map $X \to \mathsf{Phys}$, but in the subsequent observer-induced descent $\mathsf{Phys} \twoheadrightarrow Y$ once fixed reversible flow data are in place and a proper internal observer family is forward-compatible with the descended flow but fails to be backward-stable. The arrow of time is therefore derived internally rather than postulated externally.

Accordingly, chapter 11 stabilizes this dynamical picture by assembling refinement and observer-time quotients into a canonical two-parameter inverse-limit arena later used in chapters 14 and 17.

Chapter 11

Transport-Closed Dynamics and Spacetime Interleaving

11.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter determines the transport-closed dynamics consequences of items (SP4) and (SP5) on the closed side of the stack. Using the observer-side irreversible filtration inherited from chapter 10, it then assembles the canonical stitched refinement/observer arena ST used immediately in chapter 12 and later in chapters 14 and 17. Chapter 10 isolated the source of irreversibility in the closed-system stack. The intrinsic quotient

$$\pi\colon X \to \mathsf{Phys} := X/G$$

does not itself generate an arrow of time: reversible evolution on X descends to reversible evolution on Phys. Irreversibility enters only at the later internal observer-induced descent on Phys, and the observer-side temporal filtration used here is inherited from that stage. The purpose of the present chapter is different. We return to the closed side of the framework and determine what transport closure entails on admissible dynamics before any observer reduction is imposed. Two facts are proved.

(i) Under closed-system semantics and transport closure, every admissible dynamical evolution preserves the primitive comparison structure and therefore lies in the intrinsic symmetry group

$$G = \mathrm{Aut}(U, \mathcal{C}).$$

(ii) Once comparison preservation is determined, admissible dynamics acts canonically on the refinement tower of Boolean algebras and hence on its Stone inverse limit. If an observer reduction on the limit descends to finite refinement levels, then temporal coarse-graining and spatial refinement interleave functorially, yielding a canonical two-parameter inverse system.

To state the second conclusion in an intrinsically categorical form, we prove not merely that each admissible symmetry acts on each refinement stage, but that

$$\mathrm{Aut}(U, \mathcal{C})$$

acts by automorphisms of the entire refinement inverse system. The resulting two-parameter limit object is the first object in the stack in which refinement and observer-induced temporal coarsening are assembled into a single canonical construction. The logical position of the chapter is therefore

$$\text{transport closure} \Longrightarrow \text{comparison-preserving dynamics} \Longrightarrow \text{refinement action}$$
$$\Longrightarrow \text{refinement/observer interleaving} \Longrightarrow \mathsf{ST}.$$

No curvature, smooth realization, or macroscopic field law appears here. Those belong to later chapters.

11.2 Transport closure determines comparison preservation

We begin by fixing the meaning of transport closure at the present level of the stack.

Definition 11.2.1 (Transport-closed extension)**.** An extension of quotient semantics is called *transport-closed* if it introduces neither

(i) object-locus representative data, i.e. no section

$$s\colon \mathsf{Phys} \to X \qquad \text{with} \qquad \pi \circ s = \mathrm{id}_{\mathsf{Phys}},$$

nor

(ii) morphism-locus obstruction, i.e. no nontrivial holonomy or loop defect.

Equivalently, by the descent–obstruction classification together with the loop criterion from the transport spine (Theorem 4.5.1 and the endpoint-determined criterion in the descent–obstruction chapter), admissible transport is endpoint-determined.

Definition 11.2.2 (Admissible dynamics)**.** A dynamical evolution on U is called *admissible* if it is realized within a transport-closed extension of quotient semantics.

The first result removes an external hypothesis. Comparison preservation is not assumed; it is determined.

Lemma 11.2.3 (Transport-closed endpoint-determined realizations add no new comparison data)**.** *Under the standing principle Standing Principle 1, assume closed-system semantics and transport closure. Any endpoint-determined realization of quotient-level dynamics introduces no comparison distinction beyond the descended content on* Phys*.*

Proof. By the two-locus classification, any admissible enrichment beyond quotient semantics is exhausted by object-locus representative choice and morphism-locus transport obstruction, possibly with both and with no third locus. Endpoint-determined transport excludes the morphism locus, while transport closure excludes the object locus. Hence no additional comparison distinction can be introduced beyond the quotient-level content already determined by Phys. □

Theorem 11.2.4 (Transport closure determines comparison preservation)**.** *Under the standing principle Standing Principle 1, assume closed-system semantics and transport closure. Then every admissible dynamical transformation on U preserves every comparison predicate and therefore belongs to the intrinsic symmetry group*

$$G = \mathrm{Aut}(U, \mathcal{C}).$$

Proof. Let $\Phi \colon U \to U$ be admissible. Suppose

$$\Phi \notin \mathrm{Aut}(U, \mathcal{C}).$$

Then, by definition of $\mathrm{Aut}(U, \mathcal{C})$, there exist $c \in \mathcal{C}$ and $u, v \in U$ such that

$$c(\Phi(u), \Phi(v)) \neq c(u, v).$$

Under rectangular completeness, the comparison world admits its canonical diagonal presentation

$$U \cong X := X_A \times X_B, \qquad \pi \colon X \to \mathsf{Phys} := X/G,$$

and closed-system semantics identifies admissible state content with data descending through π. Thus any admissible realization of Φ must respect the quotient boundary determined by π. But the inequality

$$c(\Phi(u), \Phi(v)) \neq c(u, v)$$

shows that Φ produces a comparison distinction not already determined by the descended quotient content on Phys. By Theorem 11.2.3, no endpoint-determined realization can produce such a distinction. Hence any realization of Φ must involve enrichment beyond endpoint-determined quotient semantics. By the universal two-locus classification of enrichment (Theorem 2.6.1 in chapter 3, equivalently the action-groupoid classification together with the section–holonomy dichotomy), every such non-endpoint-determined realization is exhausted by representative choice and morphism-level transport, possibly with both and with no third locus. Transport closure excludes both. This contradiction proves that $\Phi \in \mathrm{Aut}(U, \mathcal{C})$. □

Remark 11.2.5 (Derived symmetry). Theorem 11.2.4 replaces the external hypothesis that admissible dynamics preserves comparisons. In the closed-system stack, comparison preservation is not an independent postulate. It is a consequence of admissibility together with transport closure.

11.3 Refinement tower and Stone inverse limit

We now pass from the comparison layer to the refinement layer. Let

$$(\mathsf{B}_k)_{k\in\mathbb{N}}$$

be the refinement tower of chapter 1, and let

$$\mathsf{B}_\infty := \left\langle \bigcup_{k\in\mathbb{N}} \mathsf{B}_k \right\rangle$$

be the generated limit Boolean algebra. Write

$$S_k := \mathrm{UF}(\mathsf{B}_k), \qquad S_\infty := \mathrm{UF}(\mathsf{B}_\infty)$$

for the corresponding Stone spaces. Restriction of ultrafilters along the inclusions

$$\mathsf{B}_k \subseteq \mathsf{B}_{k+1}$$

induces bonding maps

$$\phi_{k+1,k} \colon S_{k+1} \to S_k.$$

Proposition 11.3.1 (Stone inverse limit)**.** *There is a canonical homeomorphism*

$$S_\infty \cong \varprojlim_k S_k.$$

We denote the inverse-limit projections by

$$p_k \colon S_\infty \to S_k.$$

Proof. This is the inverse-limit representation of admissible ultrafilters proved in the foundational chapter, applied to the present refinement tower. □

Lemma 11.3.2 (Surjectivity of the projections)**.** *For every k, the projection*

$$p_k \colon S_\infty \to S_k$$

is surjective.

Proof. Each bonding map

$$\phi_{k+1,k} \colon S_{k+1} \to S_k$$

is induced by restriction of ultrafilters along

$$\mathsf{B}_k \subseteq \mathsf{B}_{k+1}.$$

Every ultrafilter on B_k extends to an ultrafilter on B_{k+1}, so each $\phi_{k+1,k}$ is surjective. The coordinate projections from an inverse limit of nonempty compact Hausdorff spaces with surjective bonding maps are therefore surjective. Under the identification of Theorem 11.3.1, this is exactly the surjectivity of p_k. □

11.4 Functorial action on the refinement inverse system

Once admissible dynamics is determined into $\mathrm{Aut}(U,\mathcal{C})$, it acts not only on U but on every comparison-generated Boolean stage, and hence on the full Stone tower.

Proposition 11.4.1 (Boolean invariance). *Every comparison-preserving bijection*

$$\phi \in \mathrm{Aut}(U,\mathcal{C})$$

acts on B_∞ *by pullback*

$$E \longmapsto \phi^{-1}(E),$$

preserves each finite-stage algebra B_k*, and therefore induces maps*

$$\widehat{\phi}^{(k)}: S_k \to S_k, \qquad \widehat{\phi}^{\infty}: S_\infty \to S_\infty.$$

Proof. It is enough to check preservation on the generating comparison sets. For a basic left comparison set

$$L_{c,w} := \{u \in U : c(u,w) = 1\},$$

we have

$$u \in \phi^{-1}(L_{c,w}) \iff c(\phi(u),w) = 1 \iff c(u,\phi^{-1}(w)) = 1 \iff u \in L_{c,\phi^{-1}(w)}.$$

Thus pullback sends left generators to left generators. Similarly, for a basic right comparison set

$$R_{c,w} := \{u \in U : c(w,u) = 1\},$$

one computes

$$u \in \phi^{-1}(R_{c,w}) \iff c(w,\phi(u)) = 1 \iff c(\phi^{-1}(w),u) = 1 \iff u \in R_{c,\phi^{-1}(w)}.$$

Hence right generators are preserved as well. Therefore pullback preserves the generating comparison family, hence every B_k and therefore B_∞. The induced maps on Stone spaces are the stated maps. ☐

Theorem 11.4.2 (Functorial action on the refinement inverse system). *The assignment*

$$\phi \longmapsto \left(\widehat{\phi}^{(k)}\right)_{k\in\mathbb{N}}$$

defines an action of

$$\mathrm{Aut}(U,\mathcal{C})$$

by automorphisms of the inverse system

$$\left(S_k, \phi_{k+1,k}\right)_{k\in\mathbb{N}}.$$

Equivalently, for every

$$\phi \in \mathrm{Aut}(U, \mathcal{C})$$

and every k, the square

$$\begin{array}{ccc} S_{k+1} & \xrightarrow{\widehat{\phi}^{(k+1)}} & S_{k+1} \\ {\scriptstyle \phi_{k+1,k}}\downarrow & & \downarrow{\scriptstyle \phi_{k+1,k}} \\ S_k & \xrightarrow[\widehat{\phi}^{(k)}]{} & S_k \end{array}$$

commutes, and

$$\widehat{\mathrm{id}}^{(k)} = \mathrm{id}_{S_k}, \qquad \widehat{(\phi \circ \psi)}^{(k)} = \widehat{\phi}^{(k)} \circ \widehat{\psi}^{(k)}$$

for all k.

Proof. Fix $\phi \in \mathrm{Aut}(U, \mathcal{C})$. By Theorem 11.4.1, the maps

$$\widehat{\phi}^{(k)} \colon S_k \to S_k$$

are well-defined. To verify compatibility with the bonding maps, let $\mathfrak{u} \in S_{k+1}$. Then

$$\phi_{k+1,k}\big(\widehat{\phi}^{(k+1)}(\mathfrak{u})\big) = \widehat{\phi}^{(k+1)}(\mathfrak{u}) \cap \mathsf{B}_k.$$

By definition,

$$E \in \widehat{\phi}^{(k+1)}(\mathfrak{u}) \iff \phi^{-1}(E) \in \mathfrak{u} \qquad (E \in \mathsf{B}_{k+1}).$$

Restricting to $E \in \mathsf{B}_k$, we obtain

$$E \in \phi_{k+1,k}\big(\widehat{\phi}^{(k+1)}(\mathfrak{u})\big) \iff \phi^{-1}(E) \in \mathfrak{u} \cap \mathsf{B}_k \iff E \in \widehat{\phi}^{(k)}(\mathfrak{u} \cap \mathsf{B}_k).$$

Thus

$$\phi_{k+1,k} \circ \widehat{\phi}^{(k+1)} = \widehat{\phi}^{(k)} \circ \phi_{k+1,k}.$$

Identity and composition are immediate from the corresponding facts for pullback on subsets of U. The identity automorphism induces the identity on every Boolean algebra, hence

$$\widehat{\mathrm{id}}^{(k)} = \mathrm{id}_{S_k}.$$

If $\phi, \psi \in \mathrm{Aut}(U, \mathcal{C})$, then

$$(\phi \circ \psi)^{-1} = \psi^{-1} \circ \phi^{-1},$$

so

$$\widehat{(\phi \circ \psi)}^{(k)} = \widehat{\phi}^{(k)} \circ \widehat{\psi}^{(k)}.$$

Therefore $\mathrm{Aut}(U, \mathcal{C})$ acts by automorphisms of the inverse system. ☐

Corollary 11.4.3 (Action on the Stone limit)**.** *The action of* $\mathrm{Aut}(U, \mathcal{C})$ *on the inverse system induces an action on the inverse limit S_∞. Under the identification*

$$S_\infty \cong \varprojlim_k S_k,$$

this action agrees with the Stone action

$$\widehat{\phi}^{\infty} \colon S_\infty \to S_\infty.$$

Proof. By Theorem 11.4.2, each $\phi \in \mathrm{Aut}(U, \mathcal{C})$ determines a compatible family of maps on the inverse system and hence a unique induced map on the inverse limit. By construction, this is the Stone-dual map induced directly from pullback on B_∞. $\square$

Corollary 11.4.4 (Induced flow on the Stone limit)**.** *Let*

$$(\Phi_t)_{t\in\mathbb{R}}$$

be an admissible reversible flow on U. Then each Φ_t induces compatible homeomorphisms

$$\widehat{\Phi}_t^{(k)} \colon S_k \to S_k, \qquad \widehat{\Phi}_t^{\infty} \colon S_\infty \to S_\infty,$$

and

$$(\widehat{\Phi}_t^{\infty})_{t\in\mathbb{R}}$$

is a reversible flow on S_∞.

Proof. By Theorem 11.2.4, each Φ_t lies in $\mathrm{Aut}(U, \mathcal{C})$, so Theorems 11.4.2 and 11.4.3 produce the induced maps on every S_k and on S_∞. Since

$$\Phi_{t+s} = \Phi_t \circ \Phi_s, \qquad \Phi_0 = \mathrm{id}_U,$$

the induced maps satisfy

$$\widehat{\Phi}_{t+s}^{(k)} = \widehat{\Phi}_t^{(k)} \circ \widehat{\Phi}_s^{(k)}, \qquad \widehat{\Phi}_0^{(k)} = \mathrm{id}_{S_k},$$

and likewise on S_∞. Hence $(\widehat{\Phi}_t^{\infty})_{t\in\mathbb{R}}$ is a reversible flow. $\square$

11.5 Observer reduction on the Stone limit

We now reintroduce observer structure, but only at the limit level. Let

$$\theta_\infty \colon S_\infty \twoheadrightarrow Y$$

be a surjective observer reduction.

Definition 11.5.1 (k-level admissibility)**.** The observer reduction θ_∞ is called *k-level admissible* if it is constant on fibers of

$$p_k \colon S_\infty \to S_k,$$

that is,

$$p_k(x) = p_k(y) \Longrightarrow \theta_\infty(x) = \theta_\infty(y).$$

Lemma 11.5.2 (Descent criterion)**.** *Fix k. The following are equivalent:*

(i) *There exists a map*

$$\theta_k \colon S_k \to Y$$

such that

$$\theta_k \circ p_k = \theta_\infty.$$

(ii) θ_∞ *is* k*-level admissible.*

If such a map exists, it is unique.

Proof. If θ_k exists and $p_k(x) = p_k(y)$, then

$$\theta_\infty(x) = \theta_k(p_k(x)) = \theta_k(p_k(y)) = \theta_\infty(y),$$

so θ_∞ is constant on fibers of p_k. Conversely, assume θ_∞ is constant on fibers of p_k. Given $u \in S_k$, choose $x \in S_\infty$ with $p_k(x) = u$, which is possible by Theorem 11.3.2, and define

$$\theta_k(u) := \theta_\infty(x).$$

This is well-defined by fiber constancy. The identity

$$\theta_k \circ p_k = \theta_\infty$$

is immediate, and uniqueness follows from the surjectivity of p_k. □

Definition 11.5.3 (Induced finite-level filtration)**.** Assume θ_∞ is k-level admissible, and let

$$\theta_k \colon S_k \to Y$$

be the descended map of Theorem 11.5.2. For $t \geq 0$, define

$$\mathcal{F}_t^{(k)} := \ker\bigl(\theta_k \circ \widehat{\Phi}_t^{(k)}\bigr) \subseteq S_k \times S_k,$$

that is,

$$(a, b) \in \mathcal{F}_t^{(k)} \iff \theta_k\bigl(\widehat{\Phi}_t^{(k)}(a)\bigr) = \theta_k\bigl(\widehat{\Phi}_t^{(k)}(b)\bigr).$$

Remark 11.5.4 (No independent finite-level observer)*.* Finite-level observer data are not primitive. If θ_∞ is k-level admissible, then θ_k exists uniquely. If θ_∞ is not k-level admissible, no such θ_k exists. Any independent specification of a finite-level observer map would therefore constitute additional structure beyond the closed-system boundary.

11.6 Determined interleaving across refinement

We now prove that observer-induced temporal identifications descend along refinement whenever observer descent is available at adjacent levels.

Theorem 11.6.1 (Interleaving)**.** *Assume that* θ_∞ *is both* $(k+1)$*-level admissible and* k*-level admissible. Then for every* $t \geq 0$*,*

$$(\phi_{k+1,k} \times \phi_{k+1,k})\bigl(\mathcal{F}_t^{(k+1)}\bigr) \subseteq \mathcal{F}_t^{(k)}.$$

Proof. Let
$$(u,v) \in \mathcal{F}_t^{(k+1)}.$$
Then
$$\theta_{k+1}\big(\widehat{\Phi}_t^{(k+1)}(u)\big) = \theta_{k+1}\big(\widehat{\Phi}_t^{(k+1)}(v)\big).$$
Choose $x, y \in S_\infty$ with
$$p_{k+1}(x) = u, \qquad p_{k+1}(y) = v.$$
Using
$$\theta_{k+1} \circ p_{k+1} = \theta_\infty$$
and
$$p_{k+1} \circ \widehat{\Phi}_t^{\infty} = \widehat{\Phi}_t^{(k+1)} \circ p_{k+1},$$
we obtain
$$\theta_\infty\big(\widehat{\Phi}_t^{\infty}(x)\big) = \theta_\infty\big(\widehat{\Phi}_t^{\infty}(y)\big).$$
Now use
$$p_k = \phi_{k+1,k} \circ p_{k+1}, \qquad p_k \circ \widehat{\Phi}_t^{\infty} = \widehat{\Phi}_t^{(k)} \circ p_k, \qquad \theta_k \circ p_k = \theta_\infty.$$
Then
$$\theta_k\big(\widehat{\Phi}_t^{(k)}(\phi_{k+1,k}(u))\big) = \theta_k\big(\widehat{\Phi}_t^{(k)}(\phi_{k+1,k}(v))\big),$$
which is exactly the statement that
$$(\phi_{k+1,k}(u), \phi_{k+1,k}(v)) \in \mathcal{F}_t^{(k)}.$$
□

Corollary 11.6.2 (Refinement contraction)**.** *Define*
$$d_k(a,b) := \inf\{t \geq 0 : (a,b) \in \mathcal{F}_t^{(k)}\},$$
whenever the infimum is meaningful. Then
$$d_k\big(\phi_{k+1,k}(u), \phi_{k+1,k}(v)\big) \leq d_{k+1}(u,v).$$

Proof. If
$$(u,v) \in \mathcal{F}_t^{(k+1)},$$
then Theorem 11.6.1 gives
$$(\phi_{k+1,k}(u), \phi_{k+1,k}(v)) \in \mathcal{F}_t^{(k)}.$$
Taking the infimum over all such t yields the claim. □

11.7 Spacetime as a joint two-parameter limit

Theorems 11.4.2 and 11.6.1 produce two independent directions of structure.

- Refinement yields the inverse system

$$\cdots \to S_{k+1} \to S_k \to \cdots .$$

- Observer-induced temporal coarse-graining yields, at each admissible level k, the quotient family

$$S_k/\mathcal{F}_t^{(k)}.$$

Consider therefore the two-parameter diagram

$$(k,t) \longmapsto S_k/\mathcal{F}_t^{(k)},$$

restricted to those pairs (k,t) for which θ_∞ descends to level k and for which the temporal bonding maps are defined. The refinement-direction bonding maps are supplied by Theorem 11.6.1. Temporal bonding maps exist on any subdomain of t on which

$$t \longmapsto \mathcal{F}_t^{(k)}$$

is monotone.

Definition 11.7.1 (Spacetime object)**.** The *spacetime object* is

$$\mathsf{ST} := \varprojlim_{(k,t)} \big(S_k/\mathcal{F}_t^{(k)}\big),$$

taken over the two-parameter domain on which the quotients and bonding maps are defined.

Remark 11.7.2 (Interpretation). The object ST packages two distinct structural directions:

(i) spatial refinement, represented by the inverse-system direction in k;

(ii) temporal coarsening, represented by the observer-induced quotient direction in t.

It is therefore the first object in the stack in which the two limiting directions coexist in a single canonical construction.

Remark 11.7.3 (Determined versus observer-relative data). Up to this point, the following data are determined:

- the intrinsic symmetry group

$$G = \mathrm{Aut}(U, \mathcal{C});$$

- comparison preservation of admissible dynamics under transport closure;
- the functorial action of $\mathrm{Aut}(U, \mathcal{C})$ on the refinement inverse system;
- the induced action on the Stone limit;
- the refinement-direction interleaving maps;
- the corresponding two-parameter spacetime object on every domain where observer descent and temporal monotonicity are available.

What remains observer-relative is the choice of reduction

$$\theta_\infty : S_\infty \to Y,$$

and hence the set of finite levels at which descent occurs, together with the temporal domain on which the induced filtration is monotone.

Remark 11.7.4 (Position in the stack). This chapter does not yet introduce curvature. Its output is the structural arena on which later curvature data will be realized:

$$\text{transport-closed dynamics} \Longrightarrow \mathrm{Aut}(U, \mathcal{C})\text{-action}$$
$$\Longrightarrow \text{refinement/observer interleaving} \Longrightarrow \mathsf{ST}.$$

The curvature chapters begin only after this arena has been fixed.

11.8 Conclusion

This chapter fixes the first joint refinement–time arena in the stack: the two-parameter inverse-limit object ST, where intrinsic refinement and observer-induced temporal coarsening are represented within one coherent construction.

Accordingly, chapter 12 uses the stitched arena ST to realize the first intrinsic transport obstruction as smooth curvature, while chapters 14 and 17 reuse that same arena later in the causal and connection-first developments.

Part V

Curvature and the Einstein Boundary

Chapter 12

Relational Loop Defect and Riemann Curvature

12.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the smooth-realization consequences of the transport-visibility clause item (SP5). Combining the discrete obstruction arc of chapters 7 to 9 with the stitched refinement/observer interleaving arena of chapter 11, it derives the first smooth curvature realization used in chapter 13. Chapters 7 to 9 isolated an intrinsic obstruction to endpoint-determined transport. At finite arity the first witness of that obstruction occurs on triangle boundaries. At the algebraic level, its first visible carrier lies in the quadratic commutator layer. The purpose of the present chapter is to identify the canonical smooth realization of that obstruction. Differential geometry is not introduced here as an additional axiom. The relevant obstruction has already been isolated on intrinsic grounds: comparison structure, quotient semantics, transport, loop defect, and triangle obstruction all arise before any smooth manifold is mentioned. The question is how this intrinsic obstruction appears under smooth realization. The answer is that for Levi–Civita transport on a smooth pseudo-Riemannian manifold, the leading infinitesimal loop defect around a minimal 2-cell is precisely Riemann curvature. Four ingredients enter. First, the transport group carries the augmentation filtration, and the first nontrivial commutator carrier is

$$F^2K/F^3K.$$

Second, triangle boundaries furnish the first finite witnesses of morphism-level transport defect. The first obstruction is therefore already intrinsically 2-skeletal. Third, triangle coherence is encoded by a quotient of the free path groupoid characterized by a universal property: it is the initial groupoid in which all directed triangle loops are trivial. Fourth, under Levi–Civita realization, the leading infinitesimal defect around a small 2-cell is the curvature operator

$$R(\partial_i, \partial_j).$$

The theorem proved in this chapter is that these structures match canonically: the first intrinsic transport obstruction is quadratic and 2-skeletal, and its smooth infinitesimal realization is Riemann curvature.

Remark 12.1.1 (Ontological transition). Everything prior to smooth realization is intrinsic and algebraic: comparison structure, quotient semantics, transport, loop defect, and triangle obstruction. Smooth manifolds enter only as realizations of the already isolated transport residue. Geometry is therefore not postulated independently. It is recognized as the smooth form of an intrinsic obstruction.

12.2 Augmentation filtration and the first commutator carrier

Let K be a group. Let

$$\varepsilon : \mathbb{Z}[K] \to \mathbb{Z}$$

be the augmentation homomorphism, let

$$I := \ker(\varepsilon)$$

be the augmentation ideal, and let

$$\widehat{\mathbb{Z}[K]}$$

denote the I-adic completion.

Definition 12.2.1 (Augmentation filtration)**.** For each $m \geq 1$, define

$$F^m K := \{g \in K : g - 1 \in I^m \subseteq \widehat{\mathbb{Z}[K]}\}.$$

The associated graded pieces are

$$\mathrm{gr}^m K := F^m K / F^{m+1} K.$$

Remark 12.2.2 (Degree 1 is tautological). For every $g \in K$, one has $\varepsilon(g) = 1$, hence

$$g - 1 \in I.$$

Therefore

$$F^1 K = K.$$

Thus degree 1 carries no commutator information. The first nontrivial graded transport residue can appear only in degree 2.

Lemma 12.2.3 (Commutators raise augmentation order)**.** *For all integers $r, s \geq 1$,*

$$[F^r K, F^s K] \subseteq F^{r+s} K.$$

Proof. Let $a \in F^r K$ and $b \in F^s K$. Then there exist

$$u \in I^r, \qquad v \in I^s$$

such that

$$a = 1 + u, \qquad b = 1 + v$$

inside $\widehat{\mathbb{Z}[K]}$. Since $u \in I^r$ and $v \in I^s$, the geometric-series expansions in the I-adic completion give

$$a^{-1} = 1 - u + O(I^{r+1}), \qquad b^{-1} = 1 - v + O(I^{s+1}).$$

Hence modulo I^{r+s+1},

$$\begin{aligned} aba^{-1}b^{-1} &\equiv (1+u)(1+v)(1-u)(1-v) \\ &\equiv 1 + uv - vu. \end{aligned}$$

Since $uv - vu \in I^{r+s}$, it follows that

$$[a, b] - 1 \in I^{r+s}.$$

Therefore

$$[a, b] \in F^{r+s} K.$$

□

Corollary 12.2.4 (First visible noncommutativity)**.** *In a one-object transport system, the first graded carrier of transport noncommutativity is*

$$F^2 K / F^3 K.$$

Proof. By Theorem 12.2.2, one has $F^1 K = K$. Hence every commutator lies in $F^2 K$ by Theorem 12.2.3 with $r = s = 1$. Quotienting by $F^3 K$ therefore isolates the first graded commutator residue. □

12.3 Triangle coherence and the universal triangle quotient

Theorems 7.5.2 and 9.7.2 show that the first finite witness of morphism-level obstruction occurs on triangle boundaries. We now encode that fact categorically. Let

$$\mathcal{N} = (V, E)$$

be a finite directed graph, and let

$$\mathrm{Path}^{\pm}(\mathcal{N})$$

denote the free path groupoid on $\mathcal{N}$. Its objects are the vertices V, and its morphisms are reduced words in directed edges and formal inverses.

Definition 12.3.1 (Directed triangle loop (intrinsic form))**.** A *directed triangle loop* in $\mathrm{Path}^{\pm}(\mathcal{N})$ is a loop of the form

$$(p \to q)(q \to r)(r \to p)$$

arising from a directed triangle in $\mathcal{N}$. This is the same notion as Theorem 9.3.3, written intrinsically in $\mathrm{Path}^{\pm}(\mathcal{N})$.

Definition 12.3.2 (Triangle congruence)**.** Let $\sim_{\triangle}$ denote the smallest groupoid congruence on $\mathrm{Path}^{\pm}(\mathcal{N})$ such that every directed triangle loop is identified with the identity at its basepoint.

Definition 12.3.3 (Triangle quotient groupoid)**.** Define

$$\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) := \mathrm{Path}^{\pm}(\mathcal{N})/{\sim_{\triangle}},$$

and write

$$Q_{\triangle} : \mathrm{Path}^{\pm}(\mathcal{N}) \to \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$$

for the quotient functor.

Theorem 12.3.4 (Universal property of the triangle quotient)**.** *Let $\mathcal{G}$ be any groupoid and let*

$$F : \mathrm{Path}^{\pm}(\mathcal{N}) \to \mathcal{G}$$

be a functor such that for every directed triangle loop ℓ,

$$F(\ell) = \mathrm{id}_{F(p)},$$

where p is the basepoint of ℓ. Then there exists a unique functor

$$\overline{F} : \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) \to \mathcal{G}$$

such that

$$F = \overline{F} \circ Q_{\triangle}.$$

Equivalently, $\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$ *is initial among groupoids receiving a functor from* $\mathrm{Path}^{\pm}(\mathcal{N})$ *in which all directed triangle loops are trivial.*

Proof. By definition, $\sim_{\triangle}$ is the smallest groupoid congruence generated by the requirement that every directed triangle loop be an identity, and closed under composition and inversion. Since F sends each directed triangle loop to an identity and preserves composition and inversion, it is constant on $\sim_{\triangle}$-equivalence classes. Therefore it factors through the quotient:

$$F = \overline{F} \circ Q_{\triangle}$$

for a unique functor

$$\overline{F} : \mathrm{Path}^{\pm}_{\triangle}(\mathcal{N}) \to \mathcal{G}.$$

Uniqueness follows because $Q_{\triangle}$ is surjective on objects and morphisms. □

Remark 12.3.5 (Structural meaning). The triangle quotient is not merely a convenient presentation. It is the initial transport domain in which triangular 2-cell boundaries are trivialized. In that sense it is the canonical receptacle for the first 2-skeletal coherence law.

12.4 Single-object transport and based triangle defect

We now pass to the single-object transport regime. Let

$$K$$

be a group, viewed as a one-object groupoid. A transport scheme on $\mathcal{N}$ is then an assignment of an element of K to each directed edge, and by the universal property of the free path groupoid it extends uniquely to a functor

$$\mathrm{Hol} : \mathrm{Path}^{\pm}(\mathcal{N}) \to BK,$$

where BK is the one-object groupoid with automorphism group K. Fix a basepoint $p \in V$. Write

$$\Omega_p(\mathcal{N}) := \mathrm{Hom}_{\mathrm{Path}^{\pm}(\mathcal{N})}(p, p)$$

for the based loop group of the free path groupoid.

Definition 12.4.1 (Based triangle normal subgroup). Let

$$N_{\triangle,p} \trianglelefteq \Omega_p(\mathcal{N})$$

be the normal subgroup generated by all based conjugates of directed triangle loops, that is, by all loops of the form

$$\alpha\, \ell\, \alpha^{-1},$$

where ℓ is a directed triangle loop based at some vertex q and $\alpha : p \to q$ is any path in $\mathrm{Path}^{\pm}(\mathcal{N})$.

Definition 12.4.2 (Based triangle quotient loop group). Define

$$\Omega_p^{\triangle}(\mathcal{N}) := \Omega_p(\mathcal{N})/N_{\triangle,p}.$$

Lemma 12.4.3 (Based loop group of the triangle quotient). *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})}(p, p) \cong \Omega_p^{\triangle}(\mathcal{N}).$$

Proof. Passing from $\mathrm{Path}^{\pm}(\mathcal{N})$ to the quotient $\mathrm{Path}^{\pm}_{\triangle}(\mathcal{N})$ imposes exactly the congruence generated by trivializing all directed triangle loops. On the based loop group at p, this identifies precisely the normal subgroup generated by all based conjugates of such loops. Therefore the based loop group of the quotient is the quotient of $\Omega_p(\mathcal{N})$ by $N_{\triangle,p}$. □

Definition 12.4.4 (Based transport defect map). The transport functor Hol induces a group homomorphism

$$\delta_p : \Omega_p(\mathcal{N}) \to K$$

by restriction to based loops at p.

Theorem 12.4.5 (Triangle–loop identification). *The following are equivalent.*

(i) *The transport functor* Hol *factors through the triangle quotient*

$$Q_\triangle : \mathrm{Path}^\pm(\mathcal{N}) \to \mathrm{Path}^\pm_\triangle(\mathcal{N}).$$

(ii) *Every directed triangle loop is sent to the identity by* Hol.

(iii) *The subgroup $N_{\triangle,p}$ lies in the kernel of*

$$\delta_p : \Omega_p(\mathcal{N}) \to K.$$

Equivalently, Hol *fails to factor through the triangle quotient if and only if the induced map on based loops descends to a nontrivial homomorphism*

$$\Omega_p^\triangle(\mathcal{N}) \to K.$$

Proof. The equivalence of (i) and (ii) is exactly Theorem 12.3.4 specialized to the functor Hol. For (ii) $\iff$ (iii): if every directed triangle loop is sent to the identity, then every based conjugate of such a loop is also sent to the identity, since δ_p is a homomorphism. Hence the normal subgroup $N_{\triangle,p}$ lies in $\ker(\delta_p)$. Conversely, every directed triangle loop based at p belongs to $N_{\triangle,p}$, and every directed triangle loop at another basepoint becomes such after conjugation by a path from p, so $\ker(\delta_p) \supseteq N_{\triangle,p}$ implies that every directed triangle loop is sent to the identity. The final statement follows from Theorem 12.4.3: the failure of factorization is exactly the failure of δ_p to kill $N_{\triangle,p}$, equivalently the existence of a nontrivial induced map on the quotient

$$\Omega_p^\triangle(\mathcal{N}) = \Omega_p(\mathcal{N})/N_{\triangle,p}.$$

□

Remark 12.4.6 (Two-dimensionality of the first obstruction). Triangle loops are the first intrinsic 2-cell boundaries in the transport system. Their nontriviality detects the first finite failure of endpoint-determined transport. The first obstruction is therefore already 2-skeletal.

12.5 Triangle boundaries and the quadratic commutator layer

We now relate the first 2-skeletal obstruction to the quadratic commutator carrier identified above.

Lemma 12.5.1 (Triangle boundaries have trivial abelianized transport content). *Let*

$$\mathrm{ab} : \Omega_p(\mathcal{N}) \to \Omega_p(\mathcal{N})^{\mathrm{ab}}$$

be the abelianization map. Then every element of $N_{\triangle,p}$ has trivial image in the abelianized transport content relevant to degree 1.

Proof. The subgroup $N_{\triangle,p}$ is generated by based conjugates of directed triangle loops. Passing to the abelianization kills conjugation and records only additive endpoint-level transport data. But a directed triangle loop is the boundary of a 2-cell, and so carries no independent 1-dimensional transport content. Hence each generator of $N_{\triangle,p}$ has trivial image in the abelianized transport content, and therefore so does every element of $N_{\triangle,p}$. □

Corollary 12.5.2 (Triangle boundary defect is invisible in degree 1)**.** *Every defect supported on the subgroup*

$$N_{\triangle,p} \trianglelefteq \Omega_p(\mathcal{N})$$

*has trivial image in the degree–*1 *transport quotient*

$$F^1K/F^2K.$$

Proof. By Theorem 12.5.1, every element of $N_{\triangle,p}$ is invisible after passage to abelianized transport content. By Theorem 12.5.3, the quotient

$$F^1K/F^2K$$

records exactly that abelianized endpoint transport content. Hence any defect supported on $N_{\triangle,p}$ has trivial image in

$$F^1K/F^2K.$$

□

Lemma 12.5.3 (Degree–1 detects only endpoint transport data)**.** *The quotient*

$$F^1K/F^2K$$

records only abelianized transport content of K. In particular, it cannot detect a defect supported on boundaries of 2*-cells.*

Proof. Since $F^1K = K$ by Theorem 12.2.2, one has

$$F^1K/F^2K = K/F^2K.$$

By Theorem 12.2.3 with $r = s = 1$, every commutator in K lies in F^2K. Hence the quotient K/F^2K kills commutators and therefore factors through the abelianization K^{ab}. It follows that F^1K/F^2K records only abelianized endpoint-level transport data and cannot detect boundary-supported 2-cell defect. □

Lemma 12.5.4 (Skeletal degree bound)**.** *Let a transport defect be detected by loops supported on boundaries of* 2*-cells. Then its first nontrivial image in the augmentation filtration lies in F^2K.*

Proof. By Theorem 12.5.2, every defect supported on 2-cell boundaries has trivial image in

$$F^1K/F^2K.$$

Therefore the first filtration level at which such a defect can appear is at least F^2K. □

Theorem 12.5.5 (First visible carrier of triangle obstruction)**.** *In a closed one-object transport regime, the first visible graded carrier of triangle obstruction is the quadratic quotient*

$$F^2K/F^3K.$$

Proof. By Theorems 12.4.5 and 12.4.6, triangle obstruction is the first finite witness of non-endpoint-determined transport and is already 2-skeletal. By Theorem 12.5.4, any such defect has trivial image in

$$F^1K/F^2K$$

and first possible nontrivial image in F^2K. By Theorem 12.2.4, the first graded carrier of transport noncommutativity is precisely

$$F^2K/F^3K.$$

Hence the first visible graded carrier of triangle obstruction is

$$F^2K/F^3K.$$

□

Remark 12.5.6 (Decategorified shadow). The quotient

$$F^2K/F^3K$$

is the first decategorified shadow of the first nontrivial 2-dimensional transport obstruction. The triangle quotient provides the 2-skeletal source; the augmentation filtration provides the first graded algebraic image.

12.6 Smooth realization by Levi–Civita transport

We now pass to smooth realization. Let (M, g) be a smooth pseudo-Riemannian manifold with Levi–Civita connection ∇.

Lemma 12.6.1 (Local geodesically convex neighborhoods)**.** *For every $x \in M$ there exists an open neighborhood U_x such that any two sufficiently near points of U_x are joined by a unique geodesic lying entirely in U_x.*

Proof. Choose normal coordinates at x. For a sufficiently small star-shaped neighborhood of $0 \in T_xM$, the exponential map is a diffeomorphism onto its image. In these coordinates geodesics are the images of radial straight lines in T_xM. Shrinking the image if necessary yields the required neighborhood. □

Lemma 12.6.2 (Finite convex witness cover)**.** *Let $C \subseteq M$ be compact. Then C admits a finite open cover*

$$\mathcal{U} = \{U_a\}_{a \in A}$$

such that each U_a is geodesically convex and every nonempty finite intersection is geodesically convex after refinement.

Proof. Choose neighborhoods U_x as in Theorem 12.6.1 for each $x \in C$. Compactness gives a finite subcover. Since only finitely many intersections occur, the sets may be shrunk simultaneously so that every nonempty finite intersection lies in a convex normal neighborhood and is therefore geodesically convex. ☐

Definition 12.6.3 (Directed witness network)**.** Fix a compact region $C \subseteq M$ and choose a finite cover

$$\mathcal{U} = \{U_a\}_{a \in A}$$

as in Theorem 12.6.2. Choose witness points

$$p_a \in U_a \qquad (a \in A),$$

and define the directed nerve network

$$\mathcal{N}(\mathcal{U}) = (V, E)$$

by

$$V := \{p_a : a \in A\}, \qquad (a \to b) \in E \iff U_a \cap U_b \neq \varnothing,$$

with orientation determined by a fixed total order on A.

Definition 12.6.4 (Levi–Civita edge transport)**.** For each directed edge $a \to b$ in the witness network, let

$$\tau_{ab} : T_{p_a}M \to T_{p_b}M$$

be parallel transport along the unique geodesic in

$$U_a \cap U_b$$

joining p_a to p_b. These edge transports extend uniquely to a transport functor

$$\mathrm{Hol}_{\mathrm{LC}} : \mathrm{Path}^{\pm}(\mathcal{N}(\mathcal{U})) \to \mathcal{C}^{\mathrm{LC}},$$

where $\mathcal{C}^{\mathrm{LC}}$ is the restriction of the frame groupoid of M to the witness vertices.

12.7 Gauge invariance of the smooth loop defect

Fix a basepoint $p \in V$. The smooth loop defect is the homomorphism

$$\delta_p^{\mathrm{LC}} : \Omega_p(\mathcal{N}(\mathcal{U})) \to \mathrm{Aut}(T_pM).$$

Lemma 12.7.1 (Gauge conjugation of loop defect)**.** *Let*

$$h_q \in \mathrm{GL}(T_qM) \qquad (q \in V)$$

be a change of local frames. Then the transformed loop defect at p satisfies

$$\delta_p^{\mathrm{LC}\prime}(\gamma) = h_p\, \delta_p^{\mathrm{LC}}(\gamma)\, h_p^{-1} \qquad \textit{for all } \gamma \in \Omega_p(\mathcal{N}(\mathcal{U})).$$

Proof. Under a frame change, each edge transport becomes

$$\tau'_{ab} = h_b\, \tau_{ab}\, h_a^{-1}.$$

For a based loop

$$\gamma = e_n \cdots e_1$$

at p, the transformed holonomy is the product of these transformed edge maps. The intermediate frame changes cancel telescopically, leaving only the basepoint terms:

$$\delta_p^{\mathrm{LC}\,\prime}(\gamma) = h_p\, \delta_p^{\mathrm{LC}}(\gamma)\, h_p^{-1}.$$

□

Corollary 12.7.2 (Intrinsic smooth defect class)**.** *The conjugacy class*

$$[\delta_p^{\mathrm{LC}}]$$

is independent of the choice of local frames. Hence smooth loop defect determines an intrinsic conjugacy-class-valued transport invariant.

Proof. Immediate from Theorem 12.7.1. □

12.8 Curvature as infinitesimal 2-cell defect

We now compute the infinitesimal form of the smooth loop defect.

Proposition 12.8.1 (Curvature commutator identity)**.** *Let ∇ be the Levi–Civita connection. Then for every vector field V,*

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)V = R(\partial_i, \partial_j)V.$$

Proof. Since the Levi–Civita connection is torsion-free, the coordinate vector fields satisfy

$$[\partial_i, \partial_j] = 0.$$

By definition of curvature,

$$R(\partial_i, \partial_j)V = \nabla_i \nabla_j V - \nabla_j \nabla_i V - \nabla_{[\partial_i, \partial_j]} V,$$

and the final term vanishes. □

Corollary 12.8.2 (Infinitesimal holonomy expansion)**.** *Let $\Sigma_{ij}(\varepsilon)$ be an infinitesimal coordinate square of side length ε based at p. Then*

$$\mathrm{Hol}(\partial \Sigma_{ij}(\varepsilon)) = \mathrm{id} + \varepsilon^2\, R(\partial_i, \partial_j)\big|_p + O(\varepsilon^3).$$

Proof. Parallel transport along a coordinate edge in the i-direction has the expansion

$$P_i = \mathrm{id} - \varepsilon\Gamma_i + O(\varepsilon^2),$$

where Γ_i denotes the Christoffel matrix in the i-direction. Multiplying the four edge transports around the oriented square, the linear terms cancel. The coefficient of ε^2 is

$$\partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i,$$

which is exactly the matrix of

$$R(\partial_i, \partial_j).$$

This gives the stated expansion. □

Lemma 12.8.3 (Filtration degree and holonomy order)**.** *Under smooth realization, the degree–m augmentation filtration corresponds to the order–m term in the small-loop holonomy expansion. In particular, modulo F^3K, only the quadratic infinitesimal holonomy term survives.*

Proof. Let γ_ε be a family of sufficiently small loops based at a fixed point, with size parameter $\varepsilon \to 0$. In a local trivialization, the corresponding holonomy admits an asymptotic expansion

$$\mathrm{Hol}(\gamma_\varepsilon) = \mathrm{id} + A_1(\varepsilon) + A_2(\varepsilon) + A_3(\varepsilon) + \cdots,$$

where $A_m(\varepsilon)$ is homogeneous of order m in ε. The augmentation filtration records the first nontrivial order at which deviation from the identity appears: membership in F^mK means that the holonomy differs from the identity only from order m onward. Hence passage to the graded quotient

$$F^mK/F^{m+1}K$$

isolates the order–m term in the small-loop expansion. For $m = 2$, modulo F^3K only the quadratic infinitesimal holonomy term survives. Thus

$$F^2K/F^3K$$

identifies with the leading quadratic holonomy operator. □

Corollary 12.8.4 (Quadratic holonomy survives modulo F^3K)**.** *Under smooth realization, the image of a sufficiently small loop in*

$$F^2K/F^3K$$

is represented exactly by its quadratic holonomy term.

Proof. This is the specialization of Theorem 12.8.3 to $m = 2$. Modulo F^3K, all terms of order 3 and higher are discarded, and the quadratic term is the first surviving contribution. □

Proposition 12.8.5 (Degree–2 graded defect and infinitesimal holonomy)**.** *Under smooth realization, the degree–2 graded piece*

$$F^2K/F^3K$$

identifies with the leading infinitesimal holonomy operators around small 2*-cells.*

Proof. By Theorem 12.8.4, the quotient

$$F^2K/F^3K$$

is represented under smooth realization by the quadratic term in the small-loop holonomy expansion. For a small 2-cell, that quadratic term is exactly the leading infinitesimal holonomy operator around the cell. ☐

Proposition 12.8.6 (Christoffel integrability coefficients)**.** *Let g be a smooth metric with Levi–Civita connection. Then Christoffel's integrability coefficients coincide with the fully lowered Riemann curvature tensor:*

$$(gkhi) = R_{gkhi}.$$

Proof. Christoffel's integrability coefficients are precisely the coordinate curvature coefficients of the Levi–Civita connection with indices lowered by the metric. Substituting the Levi–Civita coefficients into the coordinate curvature formula and lowering the remaining index gives the identity. ☐

12.9 Relational loop defect realizes as curvature

We now combine the intrinsic algebraic carrier, the universal triangle quotient, and the smooth transport computation.

Theorem 12.9.1 (Relational loop defect realizes as curvature)**.** *Under the standing principle Standing Principle 1, assume the Levi–Civita realization of Theorem 12.6.4. Then the infinitesimal relational loop defect is precisely Riemann curvature. More explicitly, for infinitesimal* 2*-cells the leading nontrivial term of loop defect is*

$$R(\partial_i, \partial_j),$$

and Christoffel's integrability coefficients are the corresponding lowered curvature components.

Proof. Intrinsic transport theory identifies morphism-level obstruction with nontrivial based loop defect. By Theorem 12.3.4, the triangle quotient is the initial transport domain in which all directed triangle loops vanish. Hence the first nontrivial finite obstruction is canonically 2-skeletal. By Theorem 12.5.5, the first visible graded carrier of that obstruction is the quadratic quotient

$$F^2K/F^3K.$$

By Theorem 12.8.5, under smooth realization this graded carrier identifies with the leading infinitesimal holonomy operators around small 2-cells. In the Levi–Civita realization, the latter are computed by Theorem 12.8.2, whose leading nontrivial term is

$$R(\partial_i, \partial_j).$$

Therefore the smooth realization of the intrinsic infinitesimal 2-cell defect is exactly Riemann curvature. The fully lowered form is Christoffel's integrability tensor, which coincides with the lowered Riemann tensor by Theorem 12.8.6. □

Remark 12.9.2 (Jet compression versus holonomy detection). Curvature can be extracted in two distinct but compatible ways. First, holonomy detects curvature directly as transport defect around a small 2-cell; this is the route intrinsic to the transport theory. Second, normal-coordinate jet expansion compresses the same curvature information into the quadratic coefficient of the metric. The familiar factor $\frac{1}{3}$ belongs to the second route, not the first: it arises from symmetrization of the metric 2-jet, not from the basic holonomy defect itself.

Remark 12.9.3 (Structural summary). This chapter identifies one transport-obstruction package through four linked descriptions:

$$\begin{aligned}\text{triangle obstruction} &\Longrightarrow \text{first 2-skeletal transport defect}\\ &\Longrightarrow \text{degree–2 graded commutator carrier}\\ &\Longrightarrow \text{smooth infinitesimal realization as Riemann curvature.}\end{aligned}$$

Equivalently, Riemann curvature is the smooth infinitesimal realization of the first intrinsic transport noncommutativity, not a separate geometric input added from outside the closed transport chain.

Remark 12.9.4 (Position in the stack). This chapter identifies curvature as the smooth realization of the first intrinsic transport obstruction. It does not yet pass to the stabilized quadratic carrier, the interface results (Theorems 13.10.4 and 13.13.1), or the Einstein boundary analysis (section 16.2). Those belong to the subsequent chapters.

12.10 Conclusion

The chapter's conclusion is exact: the first intrinsic noncommutative transport defect, carried by F^2/F^3, has Riemann curvature as its smooth infinitesimal realization, with Christoffel's integrability coefficients as its lowered coordinate form.

Accordingly, chapter 13 proves that the intrinsic quadratic carrier is the first stable obstruction on the Stone limit and that, under faithful smooth realization on that forced carrier, a nonzero stabilized square class is equivalent to nonzero realized curvature.

Chapter 13

Intrinsic 2-Skeleton Obstruction on the Stone Limit

13.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the inverse-limit interface consequences of items (SP2) and (SP5). Taking the triangle boundary and curvature realization from chapters 7 and 12 as fixed input, it proves that the first stable intrinsic transport carrier is the quadratic carrier, stabilizes that carrier on the Stone limit, and then relates nonzero stabilized square classes to nonzero realized curvature for chapters 14 to 16 under faithful smooth realization on this already-forced carrier. Chapters 7 and 12 established two structural facts (Theorems 7.5.2, 12.2.4 and 12.9.1). First, the first finite witness of morphism-level transport defect is triangular. Triangle relations therefore define the intrinsic 2-skeletal boundary of the transport system. Second, the first visible graded carrier of intrinsic transport noncommutativity is the quadratic quotient

$$F^2/F^3,$$

and, under smooth realization, its smooth infinitesimal realization is Riemann curvature. The purpose of the present chapter is to connect these two facts intrinsically, before any smooth realization is imposed. The mechanism is algebraic. At each finite stage, the isotropy group carries the augmentation filtration

$$F^1K \supseteq F^2K \supseteq F^3K \supseteq \cdots ,$$

and commutator squares automatically land in the quadratic layer F^2K. Their images in

$$F^2K/F^3K$$

define canonical degree–2 transport classes. The main point is that nontrivial triangle obstruction determines the quadratic carrier, that this determination is forced rather than postulated, that compatible quadratic square classes are preserved under refinement and hence stabilize on the Stone limit, and that under smooth realization faithful on this carrier a nonzero stabilized square class is equivalent to nonzero realized curvature. The chapter is organized as follows. We first make the refinement-compatible transport system explicit at the level of based loop groups and isotropy groups. We then define the projective triangle obstruction. Next we establish quadratic cancellation and define the degree–2 defect classes. We then isolate the passage from triangle obstruction to the quadratic carrier. After that we prove that triangle-orientation reversal descends to an intrinsic involution on the stabilized quadratic carrier. Finally we establish, under smooth realization faithful on the forced quadratic carrier, that a nonzero stabilized square class is equivalent to nonzero realized curvature. Thus the index "2" in the interface is not a hypothesis: it is the theorem proved by the triangle minimality, commutator-filtration, and refinement-stabilization chain. The remaining faithfulness condition belongs only to the realization map; it says that smooth realization does not collapse the already forced degree–2 carrier.

13.2 Refinement-compatible transport

Let

$$\mathsf{B}_1 \subseteq \mathsf{B}_2 \subseteq \cdots$$

be the Boolean refinement tower, and write

$$S_k := \mathrm{UF}(\mathsf{B}_k), \qquad S_\infty \cong \varprojlim_k S_k$$

for the associated Stone spaces. For each stage k, let

$$\mathcal{N}_k = (V_k, E_k)$$

be a finite directed transport network equipped with a transport functor

$$\mathrm{Hol}_k : \mathrm{Path}^{\pm}(\mathcal{N}_k) \longrightarrow \mathsf{C}_k.$$

Fix a basepoint $p_k \in V_k$, and define

$$\Omega_{p_k}(\mathcal{N}_k) := \mathrm{Hom}_{\mathrm{Path}^{\pm}(\mathcal{N}_k)}(p_k, p_k), \qquad K^{(k)}_{p_k} := \mathrm{Aut}_{\mathsf{C}_k}(p_k).$$

The restriction of Hol_k to based loops is the defect map

$$\delta^{(k)}_{p_k} : \Omega_{p_k}(\mathcal{N}_k) \longrightarrow K^{(k)}_{p_k}.$$

Assume the refinement package provides bonding functors

$$R_{k+1,k} : \mathrm{Path}^{\pm}(\mathcal{N}_{k+1}) \to \mathrm{Path}^{\pm}(\mathcal{N}_k), \qquad Q_{k+1,k} : \mathsf{C}_{k+1} \to \mathsf{C}_k,$$

such that

$$Q_{k+1,k} \circ \mathrm{Hol}_{k+1} = \mathrm{Hol}_k \circ R_{k+1,k}. \tag{13.2.1}$$

Lemma 13.2.1 (Bonding maps on based loops and isotropy)**.** *The refinement functors induce homomorphisms*

$$R^{\Omega}_{k+1,k} : \Omega_{p_{k+1}}(\mathcal{N}_{k+1}) \longrightarrow \Omega_{p_k}(\mathcal{N}_k), \qquad Q^{K}_{k+1,k} : K^{(k+1)}_{p_{k+1}} \longrightarrow K^{(k)}_{p_k},$$

and these satisfy

$$Q^{K}_{k+1,k} \circ \delta^{(k+1)}_{p_{k+1}} = \delta^{(k)}_{p_k} \circ R^{\Omega}_{k+1,k}.$$

Proof. Since $R_{k+1,k}$ is a functor, it sends loops at p_{k+1} to loops at p_k, hence restricts to a homomorphism

$$R^{\Omega}_{k+1,k} : \Omega_{p_{k+1}}(\mathcal{N}_{k+1}) \to \Omega_{p_k}(\mathcal{N}_k).$$

Likewise $Q_{k+1,k}$ restricts on isotropy groups to

$$Q^{K}_{k+1,k} : \mathrm{Aut}_{\mathsf{C}_{k+1}}(p_{k+1}) \to \mathrm{Aut}_{\mathsf{C}_k}(p_k).$$

Now restrict equation (13.2.1) to $\Omega_{p_{k+1}}(\mathcal{N}_{k+1})$. This yields

$$Q^{K}_{k+1,k} \circ \delta^{(k+1)}_{p_{k+1}} = \delta^{(k)}_{p_k} \circ R^{\Omega}_{k+1,k},$$

as claimed. □

13.3 Triangle obstruction

For each stage k, let

$$N^{(k)}_{\triangle,p_k} \trianglelefteq \Omega_{p_k}(\mathcal{N}_k)$$

denote the normal closure of all based triangle loops.

Definition 13.3.1 (Finite-stage triangle obstruction)**.** Define

$$\mathrm{Obs}^{(k)}_{\triangle} := \delta^{(k)}_{p_k}\big(N^{(k)}_{\triangle,p_k}\big) \subseteq K^{(k)}_{p_k}.$$

Proposition 13.3.2 (Projective triangle obstruction)**.** *The groups* $\mathrm{Obs}^{(k)}_{\triangle}$ *form a projective system under refinement. Consequently*

$$\mathrm{Obs}^{\infty}_{\triangle} := \varprojlim_{k} \mathrm{Obs}^{(k)}_{\triangle}$$

is canonically defined on the inverse-limit locus.

Proof. Refinement preserves triangle incidence, hence sends a based triangle loop at stage $k+1$ to a based triangle loop at stage k. Since $N^{(k)}_{\triangle,p_k}$ is the normal closure of the based triangle loops at stage k, it follows that

$$R^{\Omega}_{k+1,k}\big(N^{(k+1)}_{\triangle,p_{k+1}}\big) \subseteq N^{(k)}_{\triangle,p_k}.$$

Applying Theorem 13.2.1 gives

$$Q^K_{k+1,k}\Big(\delta^{(k+1)}_{p_{k+1}}\big(N^{(k+1)}_{\triangle,p_{k+1}}\big)\Big) = \delta^{(k)}_{p_k}\Big(R^{\Omega}_{k+1,k}\big(N^{(k+1)}_{\triangle,p_{k+1}}\big)\Big) \subseteq \delta^{(k)}_{p_k}\big(N^{(k)}_{\triangle,p_k}\big).$$

That is,

$$Q^K_{k+1,k}\big(\mathrm{Obs}^{(k+1)}_{\triangle}\big) \subseteq \mathrm{Obs}^{(k)}_{\triangle}.$$

Hence the triangle obstruction groups form a projective system. □

Remark 13.3.3 (Minimal 2-skeletal character). Triangle loops are the minimal 2-cell boundaries in the intrinsic transport complex. Accordingly $\mathrm{Obs}^{(k)}_{\triangle}$ is the first obstruction that is genuinely 2-skeletal rather than merely path-theoretic.

13.4 Augmentation filtration and quadratic cancellation

Let K be a group and

$$\varepsilon : \mathbb{Z}[K] \to \mathbb{Z}$$

its augmentation homomorphism. Write

$$I := \ker(\varepsilon).$$

Definition 13.4.1 (Intrinsic augmentation filtration)**.** For $m \geq 1$, define

$$F^m K := \{g \in K : g - 1 \in I^m\}.$$

Lemma 13.4.2 (Commutators raise augmentation order)**.** *For all integers $r, s \geq 1$,*

$$[F^r K, F^s K] \subseteq F^{r+s} K.$$

Proof. Let $a \in F^r K$ and $b \in F^s K$. Then

$$a - 1 \in I^r, \qquad b - 1 \in I^s.$$

Write

$$u := a - 1 \in I^r, \qquad v := b - 1 \in I^s.$$

Then

$$a = 1 + u, \qquad b = 1 + v.$$

In the I-adic completion one has convergent geometric-series expansions

$$a^{-1} = 1 - u + u^2 - u^3 + \cdots, \qquad b^{-1} = 1 - v + v^2 - v^3 + \cdots.$$

Hence

$$a^{-1} - 1 \in I^r, \qquad b^{-1} - 1 \in I^s.$$

Now compute

$$[a,b]-1 = aba^{-1}b^{-1}-1.$$

Modulo I^{r+s}, every nonconstant term in the product

$$(1+u)(1+v)\big(1+(a^{-1}-1)\big)\big(1+(b^{-1}-1)\big)$$

contains at least one factor from I^r and at least one factor from I^s, or else lies in a strictly higher power of I. Hence

$$[a,b]-1 \in I^{r+s}.$$

Therefore

$$[a,b] \in F^{r+s}K.$$

□

Now fix a stage k and a basepoint p_k. For based loops $e_i, e_j \in \Omega_{p_k}(\mathcal{N}_k)$, define the commutator square

$$\square_{ij} := e_i e_j e_i^{-1} e_j^{-1}.$$

Proposition 13.4.3 (Quadratic cancellation)**.** *For every stage k and every commutator square $\square_{ij}$,*

$$\delta_{p_k}^{(k)}(\square_{ij}) \in F^2 K_{p_k}^{(k)}.$$

Proof. Since $\delta_{p_k}^{(k)}$ is a homomorphism,

$$\delta_{p_k}^{(k)}(\square_{ij}) = [\delta_{p_k}^{(k)}(e_i), \delta_{p_k}^{(k)}(e_j)].$$

Every group element lies in F^1K. Hence Theorem 13.4.2 with $r=s=1$ gives

$$\delta_{p_k}^{(k)}(\square_{ij}) \in F^2 K_{p_k}^{(k)}.$$

□

Lemma 13.4.4 (Primitive two-cell Magnus term)**.** *Let $a, b \in K$ be two transport germs whose degree–1 classes*

$$\bar{a}, \bar{b} \in F^1K/F^2K$$

are independent. Then the commutator square

$$[a,b] = aba^{-1}b^{-1}$$

has nonzero quadratic class

$$[a,b] \bmod F^3K \in F^2K/F^3K.$$

More precisely, in the augmentation ideal one has

$$[a,b]-1 \equiv (a-1)(b-1)-(b-1)(a-1) \pmod{I^3},$$

so the class of $[a,b]$ is the exterior product $\bar{a} \wedge \bar{b}$.

Proof. Write $u = a - 1$ and $v = b - 1$. Modulo I^3,

$$a^{-1} \equiv 1 - u + u^2, \qquad b^{-1} \equiv 1 - v + v^2.$$

Multiplying

$$(1+u)(1+v)(1-u+u^2)(1-v+v^2)$$

and discarding terms in I^3 gives

$$[a,b] \equiv 1 + uv - vu \pmod{I^3}.$$

Thus

$$[a,b] - 1 \equiv uv - vu \pmod{I^3}.$$

The antisymmetric tensor $uv - vu$ is the image of $\bar{a} \wedge \bar{b}$. Since $\bar{a}$ and $\bar{b}$ are independent, this exterior product is nonzero. Hence $[a,b] \notin F^3K$, while Theorem 13.4.2 gives $[a,b] \in F^2K$. □

Lemma 13.4.5 (Minimal triangle witnesses have nonzero area class)**.** *A primitive triangle witness in the strict triangle regime determines two first-order edge germs with nonzero exterior product in F^1K/F^2K.*

Proof. By Theorem 7.5.2 and Theorem 9.6.5, every two-edge restriction of a primitive triangle witness is coherent, while the full three-edge subsystem is not. If the degree–1 edge germs of the triangle spanned a rank–0 or rank–1 subgroup, then the full triangle would factor through the same one-dimensional first-order transport line as its two-edge restrictions. In that case pairwise coherence would determine the third edge and would extend to a coherent vertex gauge on the full triangle, contradicting the strict minimality of Theorem 7.5.2. Hence the primitive triangle witness has a rank–2 first-order span. Choosing two independent edge germs a, b in that span gives $\bar{a} \wedge \bar{b} \neq 0$. □

13.5 Quadratic defect classes

Definition 13.5.1 (Quadratic defect class)**.** For each stage k, define

$$\mathcal{A}_{ij}^{(k)}(p_k) := \left[\delta_{p_k}^{(k)}(\square_{ij})\right] \in F^2K_{p_k}^{(k)}/F^3K_{p_k}^{(k)}.$$

Proposition 13.5.2 (Alternation and graded bilinearity)**.** *The assignment*

$$(i,j) \longmapsto \mathcal{A}_{ij}^{(k)}(p_k)$$

is alternating, and commutator induces a bilinear map

$$\left(F^1K_{p_k}^{(k)}/F^2K_{p_k}^{(k)}\right) \times \left(F^1K_{p_k}^{(k)}/F^2K_{p_k}^{(k)}\right) \longrightarrow F^2K_{p_k}^{(k)}/F^3K_{p_k}^{(k)}.$$

Proof. By Theorem 13.4.2, the commutator of two degree–1 elements lands in degree 2. Therefore the commutator descends to a map

$$\big(F^1K/F^2K\big) \times \big(F^1K/F^2K\big) \to F^2K/F^3K.$$

To prove bilinearity, let $a, a', b \in F^1K$. The standard commutator identity gives

$$[aa', b] = [a, b]\,[a, b, a']\,[a', b],$$

where $[a, b, a'] = [[a, b], a']$. Since $[a, b] \in F^2K$, one has

$$[a, b, a'] \in [F^2K, F^1K] \subseteq F^3K$$

by Theorem 13.4.2. Hence modulo F^3K,

$$[aa', b] \equiv [a, b][a', b].$$

Similarly,

$$[a, bb'] \equiv [a, b][a, b'] \pmod{F^3K}.$$

Thus the induced map is bilinear on the quotients. For alternation, one has

$$[a, a] = e$$

for every a, so the induced class of $[a, a]$ is zero in F^2K/F^3K. Also,

$$[b, a] = [a, b]^{-1},$$

so in additive notation on the abelian quotient F^2K/F^3K,

$$\mathcal{A}^{(k)}_{ji}(p_k) = -\mathcal{A}^{(k)}_{ij}(p_k).$$

Therefore the assignment is alternating. □

13.6 Persistence of triangle obstruction in degree 2

The next step is the intrinsic passage from triangle obstruction to a nonzero quadratic carrier.

Lemma 13.6.1 (Primitive triangle defects survive quadratically)**.** *Let*

$$g \neq e$$

be a primitive transport defect arising from a triangle witness. Then there exists a refinement stage ℓ such that

$$g \notin F^3K^{(\ell)}_{p_\ell}.$$

Proof. By Theorem 13.4.5, the primitive triangle witness determines two first-order edge germs a, b with

$$\bar{a} \wedge \bar{b} \neq 0.$$

The comparison of the two opposite orderings of these germs is the associated commutator square. By Theorem 13.4.4, that square has nonzero image in

$$F^2K/F^3K.$$

Refinement compatibility carries this primitive square class to the corresponding square class at sufficiently fine stages. Therefore the refined primitive defect cannot lie in F^3 at every stage. Hence there exists ℓ with

$$g \notin F^3K_{p_\ell}^{(\ell)}.$$

□

Lemma 13.6.2 (Persistence of nontrivial triangle defect in F^2/F^3)**.** *If*

$$\mathrm{Obs}_\triangle^{(k)} \neq \{e\},$$

then, after passing if necessary to a sufficiently fine refinement stage $\ell \geq k$, the image of $\mathrm{Obs}_\triangle^{(k)}$ *in*

$$F^2K_{p_\ell}^{(\ell)}/F^3K_{p_\ell}^{(\ell)}$$

is nonzero.

Proof. The group $\mathrm{Obs}_\triangle^{(k)}$ is generated by the images of based primitive triangle loops and their conjugates. If every primitive triangle defect had trivial image, the generated obstruction group would be trivial. Since $\mathrm{Obs}_\triangle^{(k)} \neq \{e\}$, at least one primitive triangle witness has nontrivial defect.

By refinement compatibility, that primitive defect propagates to finer stages. By Theorem 13.6.1, there exists a refinement stage $\ell \geq k$ at which its refined image does not lie in

$$F^3K_{p_\ell}^{(\ell)}.$$

On the other hand, the witness is a boundary-supported 2-cell defect, so its degree–1 image vanishes by Theorem 12.5.2 in chapter 12. Its first possible nonzero image is therefore in the quadratic range. Hence its class in

$$F^2K_{p_\ell}^{(\ell)}/F^3K_{p_\ell}^{(\ell)}$$

is nonzero, and so the image of $\mathrm{Obs}_\triangle^{(k)}$ in this quotient is nonzero. □

13.7 The relevant quadratic carrier

The next object is the intrinsic degree–2 carrier generated by quadratic square classes coming from the triangle-induced transport sector.

Definition 13.7.1 (Finite-stage quadratic carrier)**.** Let

$$\mathcal{K}_k \subseteq F^2 K^{(k)}_{p_k} / F^3 K^{(k)}_{p_k}$$

denote the subgroup generated by the classes

$$\mathcal{A}^{(k)}_{ij}(p_k)$$

arising from commutator squares in the triangle-closed transport regime.

Lemma 13.7.2 (Quadratic carrier is square-generated)**.** *For each stage k, the subgroup*

$$\mathcal{K}_k \subseteq F^2 K^{(k)}_{p_k} / F^3 K^{(k)}_{p_k}$$

defined in Theorem 13.7.1 is generated by the classes

$$\mathcal{A}^{(k)}_{ij}(p_k).$$

Proof. This is immediate from the definition of $\mathcal{K}_k$. □

Remark 13.7.3 (What is retained from triangle defect)*.* The carrier $\mathcal{K}_k$ is the intrinsic degree–2 residue of the 2-skeletal transport boundary. It is exactly the portion of triangle obstruction that remains visible after quotienting away cubic and higher transport error.

13.8 Quadratic carrier determined by triangle obstruction

Theorem 13.8.1 (Triangle obstruction determines a quadratic carrier)**.** *If*

$$\mathrm{Obs}^{(k)}_{\triangle} \neq \{e\},$$

then after refinement there exist $\ell \geq k$ and indices (i, j) such that

$$\mathcal{A}^{(\ell)}_{ij}(p_\ell) \neq 0.$$

Proof. By Theorem 13.6.2, after refinement to some $\ell \geq k$, the image of $\mathrm{Obs}^{(k)}_{\triangle}$ in

$$F^2 K^{(\ell)}_{p_\ell} / F^3 K^{(\ell)}_{p_\ell}$$

is nonzero. By definition, the relevant degree–2 carrier is the subgroup $\mathcal{K}_\ell$ generated by the square classes

$$\mathcal{A}^{(\ell)}_{ij}(p_\ell).$$

Therefore at least one such class must be nonzero. □

13.9 Universal property of the quadratic carrier

Theorem 13.8.1 shows that the degree–2 quotient is the first place where intrinsic 2-skeletal obstruction becomes visible. The next proposition isolates its universal property.

Proposition 13.9.1 (Universal property of the quadratic carrier)**.** *Let*

$$\pi_{2,3} : F^2K_p \to F^2K_p/F^3K_p$$

be the quotient map. Then $\pi_{2,3}$ is initial among group morphisms from F^2K_p to abelian groups which annihilate F^3K_p. Equivalently, if

$$\phi : F^2K_p \to A$$

is a group morphism into an abelian group A with

$$F^3K_p \subseteq \ker(\phi),$$

then there exists a unique morphism

$$\widetilde{\phi} : F^2K_p/F^3K_p \to A$$

such that

$$\phi = \widetilde{\phi} \circ \pi_{2,3}.$$

Proof. This is the universal property of the quotient group F^2K_p/F^3K_p. □

Remark 13.9.2. Theorem 13.9.1 gives the precise sense in which F^2/F^3 is the canonical carrier of quadratic transport defect. Any detector defined on the quadratic layer and insensitive to cubic error factors uniquely factors through this quotient.

13.10 Stabilization on the Stone limit

Definition 13.10.1 (Stabilized quadratic carrier)**.** Define

$$\mathcal{K}_\infty := \varprojlim_k \mathcal{K}_k.$$

Theorem 13.10.2 (Stabilized quadratic defect)**.** *Every compatible family of finite-stage quadratic square classes defines an element of*

$$\mathcal{K}_\infty = \varprojlim_k \mathcal{K}_k.$$

Conversely, every element of $\mathcal{K}_\infty$ is represented by a compatible family whose k-th component lies in $\mathcal{K}_k$.

Proof. By Theorem 13.2.1, the refinement package supplies group homomorphisms $Q^K_{k+1,k} : K^{(k+1)}_{p_{k+1}} \to K^{(k)}_{p_k}$. Any group homomorphism extends linearly to a morphism of group rings preserving augmentation ideals, so $Q^K_{k+1,k}$ carries $F^2K^{(k+1)}_{p_{k+1}}$ into $F^2K^{(k)}_{p_k}$ and $F^3K^{(k+1)}_{p_{k+1}}$ into $F^3K^{(k)}_{p_k}$. Therefore it induces a homomorphism $F^2K^{(k+1)}_{p_{k+1}}/F^3K^{(k+1)}_{p_{k+1}} \to F^2K^{(k)}_{p_k}/F^3K^{(k)}_{p_k}$. Because each $\mathcal{K}_k$ is generated by the square classes $\mathcal{A}^{(k)}_{ij}(p_k)$ and refinement carries such square classes to square classes at the previous stage, these quotient maps restrict to bonding maps $\mathcal{K}_{k+1} \to \mathcal{K}_k$. By definition of inverse limit, compatible families of finite-stage square classes then determine exactly the elements of $\mathcal{K}_\infty = \varprojlim_k \mathcal{K}_k$, and every element of $\mathcal{K}_\infty$ is represented by such a compatible family. □

Definition 13.10.3 (Stabilized square class)**.** Whenever a compatible family of finite-stage square classes

$$\big(\mathcal{A}^{(k)}_{ij}(p_k)\big)_k$$

is fixed, we write its image in $\mathcal{K}_\infty$ as

$$\mathcal{A}_{ij}(p_\infty).$$

Theorem 13.10.4 (Second-layer stabilization theorem)**.** *In the closed comparison stack, the first stable nontrivial transport obstruction in the refinement tower is the stabilized quadratic carrier*

$$\mathcal{K}_\infty \simeq F^2/F^3.$$

Equivalently, a nontrivial triangle obstruction cannot first appear in degree 1*, and it cannot be supported purely in* F^3 *or in any higher filtration layer. After refinement it has a nonzero image in the degree–2 carrier, and compatible such images stabilize on the Stone limit.*

Proof. By Theorem 7.5.2 and Theorem 9.6.5, the first finite morphism-level obstruction is triangular: every two-edge restriction is coherent, while the full three-edge subsystem need not be. Hence no degree–1 transport quotient can carry this obstruction. This is the content of Theorem 12.5.2 in chapter 12: boundary-supported 2-cell defect has trivial image in F^1/F^2.

By Theorem 13.6.2, any nontrivial triangle obstruction has, after sufficiently fine refinement, a nonzero image in

$$F^2K^{(\ell)}_{p_\ell}/F^3K^{(\ell)}_{p_\ell}.$$

By Theorem 13.8.1, this nonzero image is represented by a nonzero quadratic square class

$$\mathcal{A}^{(\ell)}_{ij}(p_\ell).$$

Equivalently, the obstruction cannot first appear in degree 3 or higher: such an appearance would force the degree–2 image to vanish, contradicting the nonzero square class just obtained.

Finally, Theorem 13.10.2 shows that refinement-compatible families of these finite-stage square classes define exactly the inverse limit carrier

$$\mathcal{K}_\infty = \varprojlim_k \mathcal{K}_k.$$

Thus the first stable nontrivial transport obstruction of the refinement tower is neither degree 1 nor a hidden higher layer. It is exactly the stabilized quadratic carrier $\mathcal{K}_\infty \simeq F^2/F^3$. □

Remark 13.10.5 (What has stabilized). The object $\mathcal{K}_\infty$ is the intrinsic degree–2 transport carrier retained by the full refinement tower. It is the first stabilized obstruction visible on the Stone inverse limit.

Definition 13.10.6 (Admissible degree–2 scalar detector)**.** Fix a basepoint p. A group morphism

$$\phi : F^2K_p \to \mathbb{R}$$

is called an admissible degree–2 scalar detector if it is invariant under multiplication by elements of F^3K_p, i.e. if

$$\phi(xz) = \phi(x) \qquad \text{for all } x \in F^2K_p,\ z \in F^3K_p.$$

Theorem 13.10.7 (Kernel annihilation at the quadratic carrier)**.** *Fix a basepoint p, and let*

$$\pi_{2,3} : F^2K_p \to F^2K_p/F^3K_p$$

be the canonical quotient. Let

$$\phi : F^2K_p \to \mathbb{R}$$

be any admissible degree–2 scalar detector. Then

$$F^3K_p \subseteq \mathrm{Ker}(\phi),$$

and therefore there exists a unique morphism

$$\overline{\phi} : F^2K_p/F^3K_p \to \mathbb{R}$$

such that

$$\phi = \overline{\phi} \circ \pi_{2,3}.$$

Proof. Let $z \in F^3K_p$. Since ϕ is admissible, Theorem 13.10.6 gives

$$\phi(xz) = \phi(x) \qquad \text{for all } x \in F^2K_p.$$

Taking $x = e$, where e is the identity element of F^2K_p, yields

$$\phi(z) = \phi(e).$$

Because ϕ is a group morphism into the additive group $\mathbb{R}$, one has

$$\phi(e) = 0.$$

Hence

$$\phi(z) = 0 \qquad \text{for all } z \in F^3 K_p.$$

Therefore

$$F^3 K_p \subseteq \mathrm{Ker}(\phi).$$

Now apply Theorem 13.9.1. Since ϕ annihilates $F^3 K_p$, there exists a unique morphism

$$\overline{\phi} : F^2 K_p / F^3 K_p \to \mathbb{R}$$

such that

$$\phi = \overline{\phi} \circ \pi_{2,3}.$$

This is the required factorization. □

13.11 Orientation reversal on the quadratic carrier

Primitive triangle witnesses admit a natural orientation reversal. We show that this operation descends to the stabilized quadratic carrier.

Definition 13.11.1 (Primitive orientation reversal)**.** Let

$$\omega(p, q, r)$$

denote the defect class associated to the oriented triangle (p, q, r). Define the reversed class by

$$\mathsf{R}_{\triangle}\, \omega(p, q, r) := \omega(p, r, q).$$

Lemma 13.11.2 (Orientation reversal preserves the triangle quotient)**.** *Let $w_{\triangle}(p, q, r)$ denote the triangle boundary word. Then reversing orientation sends*

$$w_{\triangle}(p, q, r) \mapsto w_{\triangle}(p, r, q) = w_{\triangle}(p, q, r)^{-1}.$$

Hence orientation reversal preserves the normal closure of triangle relations.

Proof. By definition

$$w_{\triangle}(p, q, r) = [rp][qr][pq].$$

Reversing orientation exchanges the last two vertices and yields the inverse word. Since the triangle relations are imposed through their normal closure, the quotient group is invariant under inversion. □

Theorem 13.11.3 (Orientation involution on the quadratic carrier)**.** *Primitive triangle reversal descends to a well-defined involution*

$$\mathsf{R}_{\mathrm{or}} : \mathcal{K}_\infty \longrightarrow \mathcal{K}_\infty.$$

Moreover

$$\mathsf{R}_{\mathrm{or}}^2 = I.$$

Proof. At each finite stage, the quotient

$$F^2 K_{p_k}^{(k)} / F^3 K_{p_k}^{(k)}$$

is abelian, so inversion is a well-defined automorphism. On square classes it exchanges

$$\mathcal{A}_{ij}^{(k)}(p_k) \longleftrightarrow \mathcal{A}_{ji}^{(k)}(p_k) = -\mathcal{A}_{ij}^{(k)}(p_k).$$

By refinement compatibility, these involutions are compatible from stage to stage and therefore descend to the inverse limit $\mathcal{K}_\infty$. Applying inversion twice yields the identity, so

$$\mathsf{R}_{\mathrm{or}}^2 = I.$$

□

Remark 13.11.4. The involution R_{or} exchanges the two orientations of every primitive triangle witness. Later chapters may use the corresponding ±1-eigenspace decomposition to organize the defect sector.

13.12 Smooth realization

Assume now a smooth realization in which square-loop transport admits a small-loop parameter ε, and suppose parallel transport around a coordinate square satisfies

$$\mathrm{Hol}(\square_{ij}(\varepsilon)) = \mathrm{id} + \varepsilon^2 R(\partial_i, \partial_j) + O(\varepsilon^3).$$

Fix a limit basepoint $p_\infty \in S_\infty$, and set

$$V := T_{\iota(p_\infty)} M.$$

Definition 13.12.1 (Second-jet comparison map)**.** Define

$$\mathsf{Jet}_2 : \mathcal{K}_\infty \to \mathrm{End}(V)$$

to be the map sending the quadratic class of an intrinsic square defect to the coefficient of ε^2 in the realized holonomy expansion.

Lemma 13.12.2 (Well-definedness of the second-jet map)**.** *The map* Jet_2 *is well-defined on* $\mathcal{K}_\infty$*.*

Proof. By Theorem 12.8.3 in chapter 12, the quotient

$$F^2/F^3$$

records exactly the quadratic holonomy coefficient, while terms of order 3 and higher are killed by passage to the quotient by F^3. Hence two representatives of the same stabilized quadratic class have the same ε^2-coefficient under smooth realization. Therefore Jet_2 is well-defined. □

Theorem 13.12.3 (Curvature identification)**.** *For the stabilized quadratic class,*

$$\mathsf{Jet}_2\big(\mathcal{A}_{ij}(p_\infty)\big) = R(\partial_i, \partial_j).$$

Proof. By definition,

$$\mathcal{A}_{ij}(p_\infty)$$

is the degree–2 class of the intrinsic commutator defect of the square loop. By Theorem 13.12.1, Jet_2 extracts the coefficient of ε^2 in the realized holonomy expansion. By Theorem 12.8.2 in chapter 12, the coefficient of ε^2 in the holonomy expansion of a small square is exactly

$$R(\partial_i, \partial_j).$$

Therefore

$$\mathsf{Jet}_2\big(\mathcal{A}_{ij}(p_\infty)\big) = R(\partial_i, \partial_j).$$

□

13.13 Interface theorem

Theorem 13.13.1 (Intrinsic–smooth interface)**.** *Under the standing principle Standing Principle 1, the refinement tower stabilizes its first nontrivial transport obstruction at the degree–2 carrier*

$$\mathcal{K}_\infty \simeq F^2/F^3.$$

Assume only realization faithfulness on this forced carrier, i.e. that the comparison map

$$\mathsf{Jet}_2 : \mathcal{K}_\infty \to \mathrm{End}(V)$$

is injective. Then the following are equivalent:

$$\exists\, \mathcal{A}_{ij}(p_\infty) \neq 0, \qquad R \neq 0.$$

Proof. The stabilization assertion is Theorem 13.10.4. Thus the appearance of the second layer is not an additional interface assumption; it is the intrinsic output of the refinement tower. What remains to check is the faithful smooth reading of that carrier.

Suppose that

$$\mathcal{A}_{ij}(p_\infty) \neq 0.$$

Since Jet_2 is injective,

$$\mathsf{Jet}_2\big(\mathcal{A}_{ij}(p_\infty)\big) \neq 0.$$

By Theorem 13.12.3,

$$\mathsf{Jet}_2\big(\mathcal{A}_{ij}(p_\infty)\big) = R(\partial_i, \partial_j),$$

hence

$$R \neq 0.$$

Conversely, if

$$R \neq 0,$$

then for some indices i, j,

$$R(\partial_i, \partial_j) \neq 0.$$

By Theorem 13.12.3,

$$\mathsf{Jet}_2\big(\mathcal{A}_{ij}(p_\infty)\big) = R(\partial_i, \partial_j) \neq 0.$$

Since the zero class has zero second-jet coefficient by Theorem 13.12.1, this implies

$$\mathcal{A}_{ij}(p_\infty) \neq 0.$$

□

Remark 13.13.2 (Status of second-layer stabilization). The aggressive question is not whether one may assume the second layer; the answer is that one may not. The second layer is forced. The proof is the chain

$$\begin{aligned}
&\text{strict triangle minimality} \Longrightarrow \text{no degree–1 residue} \\
&\Longrightarrow \text{commutator-square defect in } F^2 \Longrightarrow \text{nonzero class in } F^2/F^3 \\
&\Longrightarrow \mathcal{K}_\infty.
\end{aligned}$$

The only remaining hypothesis is not that the tower stabilizes at degree 2, but that a chosen compatible smooth realization is faithful on the already stabilized degree–2 carrier. Under that faithful reading, Theorem 13.12.3 identifies the realized image with Riemann curvature, and the GR chain proceeds.

13.14 Einstein boundary

The present chapter identifies the stabilized quadratic square-class channel as the forced intrinsic degree–2 transport carrier and, under faithful smooth realization on that carrier, shows that nonzero such classes are equivalent to nonzero realized curvature. No dynamical equations have been assumed or derived. Macroscopic compatibility laws belong to the subsequent chapter.

13.15 Conclusion

The interface result is now closed in stabilized form: the refinement tower has a forced degree–2 first obstruction, and under faithful smooth realization on that carrier the existence of a nonzero stabilized quadratic square class is equivalent to nonzero realized curvature on the same limit object.

Accordingly, chapter 14 combines the stabilized quadratic carrier, under the faithful-realization criterion that nonzero stabilized square classes are equivalent to nonzero realized curvature, with the observer-side filtration inherited from chapter 10 and interleaved with the refinement tower in chapter 11 to derive the macroscopic causal and scaling consequences forced by the closed stack.

Chapter 14

Quadratic Necessity, Four–Dimensional Diamonds, and Causality

14.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the causal-scaling consequences of combining the stabilized quadratic carrier of chapter 13 with the irreversible filtration of chapters 10 and 11. From those two inherited structures it derives the macroscopic causal geometry used later in chapters 15 and 16. Chapters 10, 11 and 13 have already established two structural facts of the closed relational stack, and the present chapter begins from them rather than re-proving them. First, the first visible intrinsic obstruction carrier is quadratic. At the intrinsic level, this carrier is the stabilized degree–2 layer, which we denote in this chapter by

$$\mathcal{K} := \mathcal{K}_\infty \;\simeq\; F^2/F^3.$$

At the realized level, the intrinsic–smooth interface developed in chapter 13 shows that the second layer is forced intrinsically and that, under faithful realization on this carrier, nonzero stabilized square classes are equivalent to nonzero realized curvature. Accordingly, before the present chapter begins, the stabilized quadratic carrier already comes with the realized-level interface criterion used here. More precisely, the intrinsic stabilization is Theorem 13.10.4, and the curvature identification is Theorem 13.12.3. Second, the stack inherits the observer-side monotone filtration arising from the irreversible observer descent of chapter 10; in chapter 11 that filtration is shown to interleave functorially with the refinement tower. This inherited filtration induces the causal preorder used below. The purpose of the present chapter is to extract the macroscopic consequences of these two structural facts. More precisely, we prove the following.

(i) Triangle closure leaves the quadratic layer unchanged. Consequently the first visible channel count is quadratic in the number of active generators.

(ii) The observer-side filtration determines the correct normalization at the first visible layer. Together with the quadratic channel count on the stabilized quadratic carrier, that canonical normalization yields parabolic scaling and hence four–dimensional causal-diamond growth.

(iii) Combined with the causal preorder, the first-variation pairing extracted, under the inherited second-jet faithfulness condition, from the realized degree–2 channel attached to the stabilized quadratic carrier is necessarily Lorentzian whenever that pairing is nondegenerate.

Thus, once the degree–2 obstruction and the inherited observer-side filtration are fixed, both the macroscopic diamond dimension and the causal signature of the realized system are sharply constrained.

Remark 14.1.1 (Role of the chapter). The point of the chapter is not to re-establish from first principles that the first visible obstruction is quadratic. That fact is already established in Theorems 13.10.2 and 13.10.4. What must be proved here is subtler: every finite-generator triangle-closed model compatible with the stack inherits a quadratic visible carrier, and that quadraticity, together with the inherited observer-side filtration, has rigid macroscopic consequences.

Remark 14.1.2 (Why the number 4 is encoded one layer deeper). The exponent 4 does not first arise at the level of diamond-volume counting. It is already determined by the interaction of two independent quadratic laws. The first visible graded piece is degree 2, so the visible channel count grows quadratically:

$$Q(k) \asymp k^2.$$

The filtration itself then determines linear accumulation of the first visible carrier:

$$M(T) \asymp T.$$

Hence

$$k(T) \asymp \sqrt{T}.$$

A causal diamond carries one filtration direction together with one quadratic visible carrier, so

$$\mu(D_T) \asymp T \cdot Q(k(T)) \asymp T^2.$$

Passing from filtration time T to linear refinement scale

$$L \asymp k(T) \asymp \sqrt{T}$$

therefore yields

$$\mu(D_L) \asymp L^4.$$

Accordingly, the exponent 4 is the macroscopic shadow of the quadratic channel law together with the inherited observer-side irreversible filtration, whose canonical normalization makes refinement scale parabolic relative to filtration time.

14.2 Triangle closure and the quadratic layer

Let

$$L_k = \langle e_1, \dots, e_k \rangle$$

be the free local transport group on the k active first-order edge germs. Let

$$\varepsilon : \mathbb{Z}[L_k] \to \mathbb{Z}, \qquad J_k := \ker(\varepsilon),$$

and define the group-valued augmentation filtration

$$\mathfrak{F}^m L_k := \{g \in L_k : g - 1 \in J_k^m\}.$$

Thus the first visible generator module is

$$V_k := \mathfrak{F}^1 L_k / \mathfrak{F}^2 L_k.$$

Lemma 14.2.1 (Degree–1 carrier)**.** *The classes*

$$\bar{e}_1, \dots, \bar{e}_k, \qquad \bar{e}_r := (e_r - 1) \bmod J_k^2,$$

form a basis of V_k. In particular,

$$V_k \cong \mathbb{Z}^k.$$

Proof. This is the standard degree–1 part of the Magnus expansion for the free group on k generators: modulo J_k^2, every word records only the total signed exponent of each generator. Therefore the classes of the k active edge germs form the indicated basis. □

Lemma 14.2.2 (Quadratic commutator carrier)**.** *The commutator pairing induces an isomorphism*

$$\Lambda^2(V_k) \cong \mathfrak{F}^2 L_k / \mathfrak{F}^3 L_k.$$

Under this isomorphism,

$$\bar{e}_i \wedge \bar{e}_j \longmapsto [e_i, e_j] \bmod \mathfrak{F}^3 L_k.$$

Proof. For $a, b \in L_k$, the same augmentation calculation used in Theorem 13.4.4 gives

$$[a, b] - 1 \equiv (a - 1)(b - 1) - (b - 1)(a - 1) \pmod{J_k^3}.$$

Thus commutator is alternating and bilinear on $\mathfrak{F}^1 L_k / \mathfrak{F}^2 L_k$, and it lands in $\mathfrak{F}^2 L_k / \mathfrak{F}^3 L_k$. For a free group, the degree–2 part of the group-valued augmentation filtration is the degree–2 free Lie piece; its basis is given by the basic commutators $[e_i, e_j]$, $i < j$. Hence the induced map from $\Lambda^2 V_k$ is an isomorphism. □

Lemma 14.2.3 (Primitive triangle boundary has degree–2 leading term)**.** *Let a primitive triangle witness have first-order edge classes $u, v, w \in V_k$ with boundary closure*

$$u + v + w = 0.$$

If the triangle is strict, so that $u \wedge v \neq 0$, then its based triangle holonomy has zero degree–1 image and nonzero degree–2 image, equal up to orientation to $u \wedge v$.

Proof. The degree–1 term is the signed boundary sum $u + v + w$, hence it vanishes. In the associated rational Lie algebra, the Baker–Campbell–Hausdorff expansion of the ordered boundary product has quadratic term

$$\frac{1}{2}\big([w, v] + [w, u] + [v, u]\big).$$

Using $w = -u - v$ in the degree–1 quotient, this expression reduces, up to the sign determined by orientation, to a nonzero scalar multiple of $u \wedge v$. Equivalently, in the integral Magnus filtration its quadratic class is the corresponding primitive exterior class up to orientation. A strict triangle witness has nonzero area class $u \wedge v \neq 0$, so its first nonzero group-filtration image is exactly degree 2. □

14.2.1 Interface seal inherited from the Stone-limit carrier

The preceding computation explains why the first possible intrinsic transport residue of a strict primitive triangle is quadratic. The Stone-limit statement used by the present chapter is the stronger stabilized form proved in Theorem 13.10.4: nontrivial triangle obstruction cannot be supported purely in F^3, and its first stable nontrivial image is the quadratic carrier

$$\mathcal{K} \simeq F^2/F^3.$$

Thus this chapter inherits, rather than assumes, the degree–2 carrier whose channel count and causal consequences are analyzed below.

14.3 Quadratic channel count

Theorem 14.3.1 (Quadratic channel count)**.** *For k active independent first-order generators, the first visible quadratic transport carrier has rank*

$$Q(k) := \operatorname{rank} \Lambda^2(V_k) = \binom{k}{2}.$$

In particular,

$$Q(k) \asymp k^2.$$

Proof. By the degree–1 carrier lemma, $V_k \cong \mathbb{Z}^k$. Therefore

$$\operatorname{rank} \Lambda^2(V_k) = \binom{k}{2}.$$

By Theorem 14.2.2, this exterior square is exactly the degree–2 commutator carrier $\mathfrak{F}^2 L_k/\mathfrak{F}^3 L_k$, hence it is the first visible quadratic transport carrier. □

Remark 14.3.2 (Exterior-bilinear necessity)*.* The quadratic carrier is not merely quadratic in cardinality; it is exterior-bilinear in structure:

$$\mathfrak{F}^2 L_k/\mathfrak{F}^3 L_k \cong \Lambda^2(V_k).$$

The number $\binom{k}{2}$ is therefore the rank of primitive oriented 2-cell channels determined by the active first-order generator module.

14.4 Filtration normalization and parabolic scaling

Let T denote the observer-side filtration parameter inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11. Let

$$k(T)$$

denote the activated refinement depth at observer-side filtration time T, and define the activated quadratic mass by

$$M(T) := Q(k(T)).$$

Theorem 14.4.1 (Determined first-visible normalization)**.** *There is a canonical normalization of the observer-side filtration time, inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, unique up to an overall positive multiplicative constant, for which*

$$M(T) \asymp T.$$

Equivalently, this inherited filtration time is the cumulative count of independent first-visible degree–2 transport classes.

Proof. By Theorem 14.3.1, the degree–2 visible mass at refinement depth k is

$$Q(k) = \binom{k}{2},$$

so for the activated depth $k(T)$ one has

$$M(T) = Q(k(T)).$$

Define the normalized observer-side filtration parameter by cumulative visible mass:

$$\tau(T) := M(T).$$

This canonically normalizes the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, and any other admissible normalization of this inherited filtration differs by multiplication by a positive constant. With this choice,

$$M(\tau) = \tau,$$

hence, after renaming τ as observer-side filtration time,

$$M(T) \asymp T.$$

Since $Q(k)$ counts independent first-visible degree–2 classes, the same normalization states equivalently that this observer-side filtration time is the cumulative count of these classes. □

Lemma 14.4.2 (Parabolic scaling)**.** *Under the canonical normalization of Theorem 14.4.1,*

$$k(T) \asymp \sqrt{T}.$$

Proof. By Theorem 14.3.1,

$$Q(k) \asymp k^2.$$

By Theorem 14.4.1,

$$M(T) = Q(k(T)) \asymp T.$$

Therefore

$$k(T)^2 \asymp T,$$

and hence

$$k(T) \asymp \sqrt{T}.$$

□

Remark 14.4.3 (Intrinsic origin of the temporal parameter). The parameter T is not an externally imposed time coordinate. It is the canonical normalization variable for the observer-side monotone filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11.

This inherited filtration records the observer-side irreversible information loss carried across the refinement tower, and the causal preorder introduced below is induced by this filtration structure. Its normalization is fixed internally by the compatibility of two independent growth laws:

$$M(T) \asymp T, \qquad Q(k) \asymp k^2.$$

Compatibility determines

$$k(T) \asymp \sqrt{T}.$$

Thus T is determined, up to overall scale, as the filtration coordinate that linearizes irreversible coarse-graining relative to the quadratic refinement carrier.

14.5 Diamond growth from closure and rectangular factorization

14.5.1 Causal preorder

Let Phys denote the observational quotient.

Definition 14.5.1 (Filtration preorder)**.** For $p, q \in \mathsf{Phys}$, define

$$p \preceq q$$

if at every observer-side filtration time inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, the image of p lies in the forward fiber of the image of q.

Lemma 14.5.2 (The filtration preorder)**.** *The relation $\preceq$ is a preorder. If backward-fiber separation holds, then it descends to a partial order on the observational quotient.*

Proof. Reflexivity is immediate. For transitivity, suppose $p \preceq q$ and $q \preceq r$. Then at every observer-side filtration time inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, the forward fiber of the image of p is contained in that of q, and that of q is contained in that of r. Hence the forward fiber of p is contained in that of r, so $p \preceq r$. If both $p \preceq q$ and $q \preceq p$, then these observer-side forward fibers coincide at every such filtration time. Under backward-fiber separation for that same inherited observer-side filtration, this determines observational equivalence. Thus antisymmetry holds after quotienting by observational equivalence. □

For $p \preceq q_T$, define the causal diamond

$$D(p, q_T) := \{z \in \mathsf{Phys} : p \preceq z \preceq q_T\}.$$

14.5.2 Mixed-scale typing from closure

We now remove the final free typing input from the diamond-growth argument. Recall from the closure theorem Theorem 4.5.1 that rectangular completeness determines the canonical product presentation

$$U \cong X_A \times X_B,$$

the diagonal action of

$$G = \mathrm{Aut}(U, \mathcal{C}),$$

the orbit quotient

$$\pi : X \to \mathsf{Phys} := X/G,$$

and, crucially, canonical descent: admissible state reports are exactly the coherent ones, equivalently exactly the maps factoring through π. Recall also the structural classification of quotient extensions: any non-endpoint dependence beyond quotient semantics is exhausted by exactly two and only two mechanisms:

(1) representative selection via a section $\mathsf{Phys} \to X$;

(2) morphism-level transport data not determined by endpoints.

In the closed system treated here, representative selection is not part of the admissible semantics. Accordingly, at any finite mixed scale (k, T), the only admissible extra content is transport content. At that scale the transport content decomposes into exactly two visible directions: the refinement-visible direction at depth k, and the observer-side-filtration-visible direction at time T, where T is measured in the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11. This is the precise input needed for the mixed-scale typing theorem.

Theorem 14.5.3 (Mixed-scale typing theorem)**.** *Fix a refinement depth k and observer-side filtration time T, inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11. Let $B_{k,T}$ denote the Boolean algebra of admissible events on the corresponding mixed (k,T)-quotient of the closed stack. Let*

$$B_k^{\mathrm{ref}} \subseteq B_{k,T}$$

be the Boolean subalgebra generated by events whose truth value is determined entirely by the stage-k refinement-visible carrier, and let

$$B_T^{\mathrm{fil}} \subseteq B_{k,T}$$

be the Boolean subalgebra generated by events whose truth value is determined entirely by the observer-side filtration data up to time T. Then

$$B_{k,T} = \mathrm{Bool}\big(B_k^{\mathrm{ref}} \cup B_T^{\mathrm{fil}}\big).$$

Equivalently, every admissible mixed-scale event is a Boolean combination of refinement events and observer-side filtration events.

Proof. By canonical descent in the closed system, admissible state reports factor through the quotient

$$\pi : X \to \mathsf{Phys}.$$

Hence there is no admissible state-level content arising from arbitrary representative choice in X. Now consider an admissible event at mixed scale (k,T), where T is measured in the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11. By definition, such an event is compatible with quotient semantics and functorial under that mixed refinement/observer-filtration structure. If its value were not determined by quotient-level endpoint data alone, then by the classification of quotient extensions it would have to arise from one of the two and only two enrichment loci: representative selection or morphism-level transport data. The first locus is excluded in the present closed-system semantics: introducing a section

$$\mathsf{Phys} \to X$$

would amount to adding representative structure, whereas admissibility here is exactly quotient admissibility. Therefore every admissible mixed-scale event can depend only on the second locus, namely transport data not determined by endpoints. At scale (k,T), the visible transport data has exactly two typed sources. First, there is the refinement-visible source: dependence on the active stage-k refinement carrier. Events of this type generate B_k^{ref}. Second, there is the observer-filtration-visible source: dependence on the observer-side filtration order or forward-fiber data up to time T. Events of this type generate B_T^{fil}. Since there is no third enrichment locus in the closed system, every admissible mixed-scale event is obtained by combining these two typed sources. Because admissible events form a Boolean algebra, this means precisely that

$$B_{k,T} = \mathrm{Bool}\big(B_k^{\mathrm{ref}} \cup B_T^{\mathrm{fil}}\big).$$

□

Remark 14.5.4 (What closure contributes). Theorem 14.5.3 is genuinely a consequence of closure. Closure does not merely force quotient semantics for states; it also removes representative selection as an admissible source of mixed-scale content. What remains is exhausted by transport, and at scale (k, T) the transport data is typed exactly by the refinement-visible carrier together with the observer-side filtration inherited from chapter 10 and interleaved with the refinement tower in chapter 11. Thus the mixed-scale Boolean algebra has no third independent generator locus.

14.5.3 Rectangular splitting and derived coarse multiplicativity

By Theorem 14.5.3, the mixed-scale admissible-event algebra $B_{k,T}$ is generated by the refinement-event and filtration-event subalgebras. The lower Boolean layer of the stack identifies, for any such generated pair, rectangular completeness, rectangle-algebra factorization, and bijectivity of the canonical marginal ultrafilter map. Applying that theorem to

$$\big(B_k^{\mathrm{ref}},\, B_T^{\mathrm{fil}}\big) \subseteq B_{k,T}$$

yields the mixed-scale rectangular splitting

$$B_{k,T} \cong B_k^{\mathrm{ref}} \otimes B_T^{\mathrm{fil}},$$

equivalently the bijection

$$\Sigma_{k,T} : UF(B_{k,T}) \longrightarrow UF(B_k^{\mathrm{ref}}) \times UF(B_T^{\mathrm{fil}}).$$

Definition 14.5.5 (Activated refinement carrier)**.** Fix a forward observer-side filtration ray $T \mapsto q_T$, inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11. Let $k(T)$ be the activated refinement depth. Under the mixed-scale splitting, define the activated visible refinement carrier by

$$\mathrm{Chan}_{k(T)} := \pi^{\mathrm{ref}}\big(D(p, q_T)\big) \subseteq UF(B_{k(T)}^{\mathrm{ref}}),$$

where π^{ref} is the refinement projection.

Definition 14.5.6 (Product-limit measure)**.** Fix compatible marginal measures

$$\mu_k^{\mathrm{ref}} \quad \text{on} \quad UF(B_k^{\mathrm{ref}}), \qquad \mu_T^{\mathrm{fil}} \quad \text{on} \quad UF(B_T^{\mathrm{fil}}),$$

and define a measure on $UF(B_{k,T})$ by transporting the product measure through the mixed-scale ultrafilter splitting. Assume that these measures are compatible under the bonding maps, so that they induce a limit measure μ on the inverse limit.

Lemma 14.5.7 (Product comparability of diamonds)**.** *Assume Theorem 14.5.6. Let $T \mapsto q_T$ be a forward observer-side filtration ray, inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, and let*

$k(T)$ be the activated refinement depth at observer-side filtration time T. Then there exist measurable sets

$$S_{k(T)} \subseteq UF(B^{\mathrm{ref}}_{k(T)}), \qquad R_T, R'_T \subseteq UF(B^{\mathrm{fil}}_T)$$

such that

$$S_{k(T)} \times R_T \;\subseteq\; \Sigma_{k(T),T}\big(D(p,q_T)\big) \;\subseteq\; S_{k(T)} \times R'_T$$

and

$$\mu^{\mathrm{fil}}_T(R_T) \asymp \mu^{\mathrm{fil}}_T([0,T]), \qquad \mu^{\mathrm{fil}}_T(R'_T) \asymp \mu^{\mathrm{fil}}_T([0,T]),$$

where the interval $[0,T]$ is measured in that observer-side filtration coordinate. Moreover one may take

$$S_{k(T)} = \mathrm{Chan}_{k(T)}\,.$$

Proof. Under the mixed-scale splitting for the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11,

$$\Sigma_{k(T),T} : UF(B_{k(T),T}) \longrightarrow UF(B^{\mathrm{ref}}_{k(T)}) \times UF(B^{\mathrm{fil}}_T)$$

is a bijection. Let

$$\pi^{\mathrm{ref}},\ \pi^{\mathrm{fil}}$$

denote the associated coordinate projections. Set

$$S_{k(T)} := \pi^{\mathrm{ref}}\big(D(p,q_T)\big) = \mathrm{Chan}_{k(T)}\,.$$

By definition, every point of $\Sigma_{k(T),T}(D(p,q_T))$ has refinement coordinate in $S_{k(T)}$, so

$$\Sigma_{k(T),T}(D(p,q_T)) \subseteq S_{k(T)} \times \pi^{\mathrm{fil}}\big(D(p,q_T)\big).$$

Define

$$R'_T := \pi^{\mathrm{fil}}\big(D(p,q_T)\big).$$

Because the preorder is induced by that observer-side monotone filtration, the observer-side filtration projection of a causal diamond is an interval in the observer-side filtration coordinate. More precisely, there exists an interval $I_T \subseteq UF(B^{\mathrm{fil}}_T)$ with

$$I_T = \pi^{\mathrm{fil}}\big(D(p,q_T)\big) = R'_T.$$

By the normalization of this inherited observer-side forward ray, this interval has observer-side filtration size comparable to the interval from 0 to T:

$$\mu^{\mathrm{fil}}_T(R'_T) = \mu^{\mathrm{fil}}_T(I_T) \asymp \mu^{\mathrm{fil}}_T([0,T]).$$

Now choose any measurable subinterval $R_T \subseteq R'_T$ satisfying

$$\mu^{\mathrm{fil}}_T(R_T) \asymp \mu^{\mathrm{fil}}_T(R'_T) \asymp \mu^{\mathrm{fil}}_T([0,T]).$$

Since $S_{k(T)}$ is by definition the full refinement projection of the diamond, every refinement coordinate in $S_{k(T)}$ occurs along that observer-side filtration ray, and shrinking only in the observer-side filtration coordinate yields

$$S_{k(T)} \times R_T \subseteq \Sigma_{k(T),T}\big(D(p, q_T)\big) \subseteq S_{k(T)} \times R'_T.$$

This proves the required product comparability. □

Theorem 14.5.8 (Derived coarse multiplicativity)**.** *Assume Theorem 14.5.6. Let $T \mapsto q_T$ be a forward observer-side filtration ray, inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, and let $k(T)$ be the activated refinement depth measured against that inherited observer-side filtration. Then there exist constants $c, C > 0$ such that*

$$c\, \mu_T^{\mathrm{fil}}([0,T])\, \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}) \le \mu(D(p, q_T)) \le C\, \mu_T^{\mathrm{fil}}([0,T])\, \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}).$$

Under the normalization conventions

$$\mu_T^{\mathrm{fil}}([0,T]) \asymp T, \qquad \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}) \asymp Q(k(T)),$$

where $[0,T]$ is measured in that observer-side filtration, this yields

$$\mu(D(p, q_T)) \asymp T\, Q(k(T)).$$

Proof. By the mixed-scale rectangular splitting for the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11,

$$B_{k(T),T} \cong B_{k(T)}^{\mathrm{ref}} \otimes B_T^{\mathrm{fil}},$$

equivalently the marginal ultrafilter map

$$\Sigma_{k(T),T} : UF(B_{k(T),T}) \longrightarrow UF(B_{k(T)}^{\mathrm{ref}}) \times UF(B_T^{\mathrm{fil}})$$

is bijective. By construction of the product-limit measure on this inherited observer-side filtration, μ is the pushforward of

$$\mu_{k(T)}^{\mathrm{ref}} \otimes \mu_T^{\mathrm{fil}}$$

through $\Sigma_{k(T),T}^{-1}$. By Theorem 14.5.7, the image of the diamond is sandwiched between two product sets with identical refinement factor and comparable observer-side filtration factors:

$$\mathrm{Chan}_{k(T)} \times R_T \subseteq \Sigma_{k(T),T}\big(D(p, q_T)\big) \subseteq \mathrm{Chan}_{k(T)} \times R'_T,$$

with

$$\mu_T^{\mathrm{fil}}(R_T) \asymp \mu_T^{\mathrm{fil}}([0,T]), \qquad \mu_T^{\mathrm{fil}}(R'_T) \asymp \mu_T^{\mathrm{fil}}([0,T]),$$

where $[0,T]$ is taken in that observer-side filtration. Applying the product measure and transporting back through $\Sigma_{k(T),T}^{-1}$ gives

$$\mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)})\, \mu_T^{\mathrm{fil}}(R_T) \le \mu(D(p, q_T)) \le \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)})\, \mu_T^{\mathrm{fil}}(R'_T).$$

Using the comparability of R_T and R'_T with $[0, T]$ in that inherited observer-side filtration, we obtain constants $c, C > 0$ such that

$$c\, \mu_T^{\mathrm{fil}}([0,T])\, \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}) \le \mu(D(p, q_T)) \le C\, \mu_T^{\mathrm{fil}}([0,T])\, \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}).$$

Under the normalization conventions

$$\mu_T^{\mathrm{fil}}([0,T]) \asymp T, \qquad \mu_{k(T)}^{\mathrm{ref}}(\mathrm{Chan}_{k(T)}) \asymp Q(k(T)),$$

this becomes

$$\mu(D(p, q_T)) \asymp T\, Q(k(T)).$$

□

Definition 14.5.9 (Diamond dimension)**.** For a forward ray $T \mapsto q_T$ in the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, define

$$\dim_\diamond := \lim_{T\to\infty} \frac{\log \mu(D(p, q_T))}{\log k(T)},$$

where $k(T)$ is the activated refinement depth along that inherited observer-side filtration ray, whenever the limit exists.

Theorem 14.5.10 (Four-dimensional diamond growth)**.** *Assume Theorems 14.4.2 and 14.5.8. Let $T \mapsto q_T$ be a forward ray in the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, and let $k(T)$ be the corresponding activated refinement depth. Then*

$$\mu(D(p, q_T)) \asymp T^2.$$

Equivalently, writing

$$L := k(T) \asymp \sqrt{T},$$

along that inherited observer-side filtration ray, one has

$$\mu(D(p, q_T)) \asymp L^4.$$

In particular,

$$\dim_\diamond = 4.$$

Proof. By Theorem 14.5.8, along a forward ray $T \mapsto q_T$ in the observer-side filtration inherited from the irreversible observer descent of chapter 10 and interleaved with the refinement tower in chapter 11, with corresponding activated refinement depth $k(T)$, one has

$$\mu(D(p, q_T)) \asymp T\, Q(k(T)).$$

By Theorem 14.3.1, the quadratic channel count along that same inherited observer-side filtration ray satisfies

$$Q(k(T)) \asymp k(T)^2.$$

By Theorem 14.4.2, the canonical normalization of that inherited observer-side filtration gives

$$k(T)^2 \asymp T.$$

Therefore

$$\mu(D(p, q_T)) \asymp T \cdot T = T^2.$$

Now set

$$L := k(T) \asymp \sqrt{T}.$$

along that inherited observer-side filtration ray. Then

$$T \asymp L^2,$$

and consequently

$$\mu(D(p, q_T)) \asymp T^2 \asymp (L^2)^2 = L^4.$$

Taking logarithms in the definition of $\dim_\diamond$ along that same inherited observer-side filtration ray gives

$$\dim_\diamond = 4.$$

□

Remark 14.5.11 (Why 4 is determined). The exponent 4 records two inherited quadratic inputs: the quadratic channel law on the stabilized visible carrier, and the observer-side filtration whose canonical normalization makes refinement scale parabolic relative to filtration time. The first is algebraic:

$$Q(k) \asymp k^2.$$

The second quadraticity is kinematic only through the inherited observer-side filtration: its canonical normalization, interleaved with the refinement tower, forces

$$T \asymp k(T)^2.$$

Combining the quadratic channel law with that inherited observer-side normalization yields

$$\mu(D_T) \asymp T \cdot Q(k(T)) \asymp T \cdot T = T^2$$

along the observer-side filtration, and therefore, after reparameterizing by the corresponding refinement scale $L := k(T)$,

$$\mu(D_L) \asymp L^4.$$

Thus the four-dimensional diamond law is the macroscopic shadow of the quadratic channel law together with the inherited observer-side irreversible filtration, whose canonical normalization makes refinement scale parabolic relative to filtration time.

14.6 Lorentzian signature from the causal preorder

Section 14.5 identifies a 4-dimensional first-variation arena from the stabilized quadratic carrier together with the inherited observer-side filtration and the macroscopic diamond-scaling law they force. The first-variation spaces themselves belong to the reconstruction stack developed later in the manuscript; what must be proved here is that the causal preorder already determines the appropriate cone structure, and hence determines Lorentzian signature on the first-variation pairing extracted, under the inherited second-jet faithfulness condition, from the realized degree–2 channel attached to the stabilized quadratic carrier.

Definition 14.6.1 (First-variation pairing)**.** For μ-almost every $p \in S_\infty$, let

$$\mathsf{V}_p$$

denote the canonical 4-dimensional first-variation space attached to p, and let

$$B_p : \mathsf{V}_p \times \mathsf{V}_p \to \mathbb{R}$$

denote the symmetric bilinear first-variation pairing extracted, under the inherited second-jet faithfulness condition, from the realized degree–2 channel attached to the stabilized quadratic carrier. Write

$$Q_p(v) := B_p(v, v).$$

Remark 14.6.2 (Logical status of V_p and B_p)*.* The existence of the first-variation spaces V_p and of the pairings B_p is not an additional hypothesis of the present chapter. It belongs to the later theorem-level reconstruction of the stack from the stabilized quadratic carrier together with, under the inherited second-jet faithfulness condition, the realized degree–2 channel attached to that carrier. The work of the present section is to derive, from the observer-side filtration preorder together with, under the inherited second-jet interface hypothesis, the causal organization of the negative directions of Q_p on the realized degree–2 channel attached to the stabilized quadratic carrier.

14.6.1 Cone structure determined by the preorder

Definition 14.6.3 (Forward and backward first-variation cones)**.** For μ-almost every $p \in S_\infty$, define

$$\mathsf{C}_p^+ \subseteq \mathsf{V}_p$$

to be the set of first-variation directions represented by forward filtration transport at p, and define

$$\mathsf{C}_p^- := -\mathsf{C}_p^+.$$

Theorem 14.6.4 (Causal cone compatibility)**.** *For μ-almost every $p \in S_\infty$, the preorder-induced future cone*

$$\mathsf{C}_p^+ \subseteq \mathsf{V}_p$$

satisfies the following properties:

(C1) C_p^+ *is a proper convex cone containing no line;*

(C2) *for every* $v \in \mathsf{C}_p^+$,

$$Q_p(v) \le 0;$$

(C3) *there exists* $v \in \mathsf{C}_p^+$ *with*

$$Q_p(v) < 0;$$

(C4) C_p^+ *contains two linearly independent vectors;*

(C5) *every vector with*

$$Q_p(v) < 0$$

lies in

$$\mathsf{C}_p^+ \cup \mathsf{C}_p^-,$$

and

$$\mathsf{C}_p^+ \cap \mathsf{C}_p^- = \{0\}.$$

Proof. We prove the five statements in order. *Proof of* (C1)*.* By construction, C_p^+ consists of first-variation directions represented by forward filtration transport. Since the filtration is monotone, composition of forward transport with forward transport is again forward. Likewise, rescaling a forward infinitesimal variation by a nonnegative scalar preserves its forward orientation. Thus C_p^+ is a convex cone. It is proper because the filtration is irreversible: if a nonzero vector v and its negative $-v$ both belonged to C_p^+, then the corresponding infinitesimal transport would be both forward and backward, determining a reversible local step and contradicting the arrow of coarsening established in Theorems 10.5.5 and 10.5.6. Hence

$$\mathsf{C}_p^+ \cap (-\mathsf{C}_p^+) = \{0\},$$

so C_p^+ contains no line. *Proof of* (C2)*.* The intrinsic–smooth interface supplies, under the second-jet faithfulness condition, the realized-level criterion relating nonzero stabilized square classes to nonzero realized curvature, and the first-visible defect in the present causal setting is measured by the quadratic first-variation pairing. Forward filtration transport is non-expansive at the first visible layer: along a forward variation, visible defect may persist or decrease, but it cannot become positive in the opposite orientation without violating monotonicity of the descent map. Therefore every forward direction satisfies

$$Q_p(v) \le 0 \qquad (v \in \mathsf{C}_p^+).$$

Proof of (C3)*.* If every forward direction satisfied $Q_p(v) = 0$, then, under the inherited second-jet faithfulness condition, the first-visible pairing on the realized degree–2 channel attached to the stabilized quadratic carrier would vanish on the forward cone. But the forward filtration is nontrivial, and the stabilized quadratic carrier is the first non-vanishing visible intrinsic obstruction layer. Hence some forward direction must carry a nonzero first-visible class, and by (C2) this forces

$$\exists\, v \in \mathsf{C}_p^+ \quad \text{such that} \quad Q_p(v) < 0.$$

Proof of (C4). By Theorem 14.5.10, causal diamonds have asymptotic growth law

$$\mu(D_L) \asymp L^4.$$

A cone generated by a single direction cannot support a genuinely four-dimensional first-variation arena: it would collapse the visible transport geometry to a one-parameter family and force a degenerate diamond law incompatible with L^4-growth. Therefore the forward cone contains at least two linearly independent directions. *Proof of* (C5). The two-locus classification theorem shows that all admissible non-endpoint transport data is exhausted by the transport locus. Within the transport locus, the monotone filtration supplies the only intrinsic orientation: forward or backward. There is no third temporal orientation compatible with the closed-system comparison semantics. Let v satisfy

$$Q_p(v) < 0.$$

Then v is a genuinely timelike first-visible direction. By the exhaustion of admissible transport loci, v must be represented by transport that is either aligned with the filtration or opposite to it. Hence

$$v \in \mathsf{C}_p^+ \cup \mathsf{C}_p^-.$$

Finally, if

$$v \in \mathsf{C}_p^+ \cap \mathsf{C}_p^-,$$

then v is simultaneously forward and backward. By irreversibility of the filtration, this is possible only for the zero direction. Therefore

$$\mathsf{C}_p^+ \cap \mathsf{C}_p^- = \{0\}.$$

This proves (C5) and completes the theorem. □

Remark 14.6.5 (Cone theorem as closure point). Theorem 14.6.4 eliminates the last local hypothesis in the Lorentzian part of the chapter. The cone structure is no longer assumed: it is determined by the observer-side filtration preorder together with, under the inherited second-jet faithfulness condition, the first-visible pairing on the realized degree–2 channel attached to the stabilized quadratic carrier, the two-locus exhaustion theorem, and the four-dimensional diamond law already proved above.

14.6.2 Null directions and non-definiteness

Lemma 14.6.6 (Null directions exist)**.** *Under the hypotheses of Theorem 14.6.4, there exists $w \neq 0$ such that*

$$Q_p(w) = 0.$$

Proof. By (C3), choose

$$v_0 \in \mathsf{C}_p^+$$

with

$$Q_p(v_0) < 0.$$

By (C4), choose

$$v_1 \in \mathsf{C}_p^+$$

linearly independent from v_0. By (C2),

$$Q_p(v_1) \leq 0.$$

If

$$Q_p(v_1) = 0,$$

then $w := v_1$ is already a nonzero null vector and we are done. Hence we may assume

$$Q_p(v_1) < 0.$$

Now consider the opposite direction

$$-v_1 \in \mathsf{C}_p^-.$$

Since $v_0 \in \mathsf{C}_p^+$ and $-v_1 \in \mathsf{C}_p^-$, the two vectors lie in opposite timelike cones. Consider the segment

$$\gamma(s) := (1-s)v_0 - sv_1, \qquad 0 \leq s \leq 1.$$

Its endpoints satisfy

$$\gamma(0) = v_0 \in \mathsf{C}_p^+, \qquad \gamma(1) = -v_1 \in \mathsf{C}_p^-.$$

We claim that the segment must leave the timelike set

$$\{v : Q_p(v) < 0\}.$$

Indeed, if

$$Q_p(\gamma(s)) < 0 \qquad \text{for all } s \in [0,1],$$

then by (C5) each $\gamma(s)$ would belong to

$$\mathsf{C}_p^+ \cup \mathsf{C}_p^-.$$

Because $[0,1]$ is connected and

$$\mathsf{C}_p^+ \cap \mathsf{C}_p^- = \{0\},$$

a continuous path of timelike vectors joining $v_0 \in \mathsf{C}_p^+$ to $-v_1 \in \mathsf{C}_p^-$ would force a crossing from the forward cone to the backward cone inside the timelike region. This is impossible without passing through the boundary separating the two sheets. Therefore there exists some $s_0 \in [0,1]$ such that

$$Q_p(\gamma(s_0)) = 0.$$

Set

$$w := \gamma(s_0).$$

Since the endpoints of the segment are nonzero and lie in disjoint timelike cones, the boundary point separating the two sheets is nonzero. Hence

$$w \neq 0 \qquad \text{and} \qquad Q_p(w) = 0.$$

This proves the existence of a nonzero null vector. □

Proposition 14.6.7 (Non-definiteness determined by causality)**.** *Under the hypotheses of Theorem 14.6.4, the form B_p is not definite. Equivalently, B_p is indefinite or degenerate.*

Proof. If B_p were positive definite or negative definite, then

$$Q_p(v) \neq 0 \qquad \text{for all } v \neq 0.$$

This contradicts Theorem 14.6.6. Therefore B_p cannot be definite. □

14.6.3 Exclusion of split signature

Lemma 14.6.8 (Split signature is incompatible with two-sheeted cones)**.** *Let Q be a nondegenerate quadratic form on a real vector space V of dimension 4 with signature $(2,2)$. Then the set*

$$\{v \in V : Q(v) < 0\}$$

is connected. In particular, it cannot be written as a disjoint union

$$\{Q < 0\} = C^+ \cup (-C^+), \qquad C^+ \cap (-C^+) = \{0\},$$

with C^+ a proper convex cone.

Proof. Choose coordinates on V in which

$$Q(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

Let

$$v = (a,b,c,d), \qquad w = (a',b',c',d')$$

be arbitrary vectors satisfying

$$Q(v) < 0, \qquad Q(w) < 0.$$

Then

$$a^2 + b^2 < c^2 + d^2, \qquad a'^2 + b'^2 < c'^2 + d'^2.$$

We first connect v to a vector whose first two coordinates vanish. Consider the path

$$v(t) := (ta, tb, c, d), \qquad t \in [0, 1].$$

Along this path,

$$Q(v(t)) = t^2(a^2 + b^2) - (c^2 + d^2).$$

Since $0 \le t^2 \le 1$, we have

$$t^2(a^2 + b^2) \le a^2 + b^2 < c^2 + d^2,$$

hence

$$Q(v(t)) < 0 \qquad \text{for all } t \in [0, 1].$$

Thus v is joined within $\{Q < 0\}$ to the vector

$$\tilde{v} := (0, 0, c, d).$$

The same argument joins w within $\{Q < 0\}$ to

$$\tilde{w} := (0, 0, c', d').$$

It therefore suffices to show that the set

$$E := \{(0, 0, u_3, u_4) : u_3^2 + u_4^2 > 0\}$$

is connected. But E identifies with

$$\mathbb{R}^2 \setminus \{0\},$$

which is path connected. Hence $\tilde{v}$ and $\tilde{w}$ are joined within E, and therefore v and w are joined within $\{Q < 0\}$. Since $v, w \in \{Q < 0\}$ were arbitrary, the set $\{Q < 0\}$ is connected. Now suppose there were a decomposition

$$\{Q < 0\} = C^+ \cup (-C^+), \qquad C^+ \cap (-C^+) = \{0\},$$

with C^+ a proper convex cone. Then C^+ and $-C^+$ would be disjoint nonempty relatively open subsets of $\{Q < 0\}$ whose union is all of $\{Q < 0\}$, contradicting connectedness. Therefore no such decomposition exists. □

Theorem 14.6.9 (Lorentzian signature is determined). *Under the standing principle Standing Principle 1, if for μ-almost every $p \in S_\infty$ the first-variation pairing B_p is nondegenerate, then*

$$\operatorname{sign}(B_p) \in \{(1, 3), (3, 1)\}.$$

Proof. A nondegenerate symmetric bilinear form on a 4-dimensional real vector space has one of the signatures

$$(4,0), \quad (3,1), \quad (2,2), \quad (1,3), \quad (0,4).$$

By Theorem 14.6.7, the definite signatures $(4,0)$ and $(0,4)$ are excluded. By Theorem 14.6.8, the split signature $(2,2)$ is excluded, because Theorem 14.6.4 gives a proper two-sheeted decomposition of the timelike set:

$$\{Q_p < 0\} = \mathsf{C}_p^+ \cup \mathsf{C}_p^-, \qquad \mathsf{C}_p^+ \cap \mathsf{C}_p^- = \{0\}.$$

Hence the only remaining possibilities are

$$(1,3) \quad \text{or} \quad (3,1).$$

□

Remark 14.6.10 (Status of the Lorentzian conclusion). The Lorentzian signature theorem is now conditional only on the nondegeneracy of the first-variation pairing itself. The cone package is no longer assumed: it has been derived from the observer-side filtration preorder together with, under the inherited second-jet faithfulness condition, the first-visible pairing on the realized degree–2 channel attached to the stabilized quadratic carrier. Thus, once nondegeneracy is granted, the causal organization needed for Lorentzian signature is already supplied by the observer-side filtration preorder together with, under the inherited second-jet faithfulness condition, the first-visible pairing on the realized degree–2 channel attached to the stabilized quadratic carrier.

14.7 Quadratic scalar laws and the Born rule

Theorems 13.10.2 and 13.10.4 identify the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3$$

as the stabilized degree–2 carrier through which the first-visible defect data considered in this section are tracked in the closed stack. By Theorem 13.10.4, a nontrivial first-visible defect cannot hide purely in F^3; accordingly the first-visible alternatives considered here are tracked through their classes in $\mathcal{K}$. In this section we derive the probability law for mutually exclusive first-visible alternatives. The derivation uses only the structural constraints already established for admissible event algebras together with the scalar-channel theory already proved for the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet faithfulness condition. The argument has four steps.

(i) first-visible alternatives are tracked through their first-visible classes in the stabilized degree–2 carrier $\mathcal{K} \simeq F^2/F^3$;

(ii) admissible branch weights are scalarizations of the realized degree–2 channel attached to that stabilized quadratic carrier under the inherited second-jet faithfulness condition;

(iii) the scalarization space of that realized channel is one-dimensional, so every admissible branch weight is proportional to its canonical scalar channel;

(iv) normalization removes the proportionality constant and yields the Born law.

14.7.1 First-visible alternatives

Definition 14.7.1 (First-visible alternatives)**.** A finite family of mutually exclusive realizable alternatives is called a family of *first-visible alternatives* if their intrinsic first-visible defect classes are represented by elements

$$v_1, \dots, v_n \in \mathcal{K}.$$

These classes encode the first visible obstruction data associated with the respective alternatives and, under the inherited second-jet faithfulness condition, the probability-bearing scalar structure is read through the realized degree–2 channel attached to the stabilized quadratic carrier.

14.7.2 Admissible branch weights

Let

$$\mathcal{A} = \{v_1, \dots, v_n\} \subset \mathcal{K}$$

be a family of first-visible alternatives. A probability assignment is a map

$$P : \mathcal{A} \to [0, 1], \qquad \sum_{i=1}^{n} P(v_i) = 1.$$

We impose the following structural conditions.

(S1) **Branch symmetry.** Probabilities are invariant under permutations of alternatives with identical intrinsic defect classes.

(S2) **Coarse-graining additivity.** If two mutually exclusive alternatives are merged into a single coarse alternative, then the probability of the coarse alternative is the sum of the probabilities of the components.

(S3) **First-visible dependence.** Probabilities depend only on the first-visible classes as tracked, under the inherited second-jet faithfulness condition, by the realized degree–2 channel attached to the stabilized quadratic carrier.

(S4) **Local order ≤ 2, naturality, and linearity.** The unnormalized branch-weight law is a carrier scalarization: under the inherited second-jet faithfulness condition it depends only on the realized degree–2 channel attached to the stabilized quadratic carrier, is natural under the realized symmetry flow, and is linear on that realized channel.

Condition (S2) is determined by the Boolean algebra of admissible events. Condition (S3) holds because no lower-order visible carrier exists and because, under the inherited second-jet interface hypothesis, the probability-bearing scalar structure is read only through the realized degree–2 channel attached to the stabilized quadratic carrier. Condition (S4) is precisely the scalar-channel structure already isolated for the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet faithfulness condition.

Definition 14.7.2 (Admissible branch weight)**.** Under the inherited second-jet faithfulness condition, an *admissible branch weight* is a map

$$W : \mathcal{K} \to \mathbb{R}_{\geq 0}$$

on the realized degree–2 channel attached to the stabilized quadratic carrier, satisfying (S1)–(S4).

14.7.3 Scalar reduction

Lemma 14.7.3 (Scalar reduction)**.** *Under conditions* (S1)–(S4)*, the probability law factors through an admissible branch weight W as*

$$P(v_i) = \frac{W(v_i)}{\sum_{j=1}^{n} W(v_j)}.$$

Proof. By branch symmetry and first-visible dependence, probabilities depend only on the first-visible classes of the alternatives as tracked in $\mathcal{K}$, while, under the inherited second-jet faithfulness condition, the probability-bearing scalar structure is read through the realized degree–2 channel attached to the stabilized quadratic carrier. By coarse-graining additivity, the probability assigned to a coarse alternative is the sum of the probabilities of its mutually exclusive constituents. Thus each alternative contributes a nonnegative scalar weight. By (S4), under the inherited second-jet interface hypothesis this scalar weight is an admissible scalarization on the realized degree–2 channel attached to the stabilized quadratic carrier, hence an admissible branch weight in the sense of Theorem 14.7.2. Normalization then yields

$$P(v_i) = \frac{W(v_i)}{\sum_j W(v_j)}.$$

□

14.7.4 Uniqueness of the scalar channel

Let

$$Q : \mathcal{K} \to \mathbb{R}_{\geq 0}$$

denote a fixed nonzero canonical scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet faithfulness condition.

Lemma 14.7.4 (Uniqueness of degree–2 scalarization)**.** *The space of admissible carrier scalarizations of the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet faithfulness condition is one-dimensional. Equivalently, if*

$$W_1, W_2 : \mathcal{K} \to \mathbb{R}$$

are admissible carrier scalarizations, then there exists

$$c \in \mathbb{R}$$

such that

$$W_1 = c\,W_2.$$

Proof. By the intrinsic–smooth interface, the scalar channel considered here is the realized degree–2 channel attached to the stabilized quadratic carrier under the second-jet faithfulness condition. In the scalar-channel theorem established later in the stack, the canonical scalar channel on that realized channel is proved to be well-defined up to an overall nonzero normalization constant because the scalar summand of the realized curvature module is one-dimensional. Equivalently, the space of admissible scalarizations of the realized degree–2 channel attached to the stabilized quadratic carrier under the second-jet faithfulness condition is one-dimensional. □

Lemma 14.7.5 (One-dimensionality of admissible branch weights)**.** *The space of admissible branch weights is one-dimensional. Equivalently, every admissible branch weight W is of the form*

$$W = c\,Q$$

for some constant $c > 0$.

Proof. By (S4), under the inherited second-jet interface hypothesis an admissible branch weight is precisely a carrier scalarization of the realized degree–2 channel attached to the stabilized quadratic carrier. By Theorem 14.7.4, the space of scalarizations of that realized channel is one-dimensional. Hence if

$$W : \mathcal{K} \to \mathbb{R}$$

is any admissible branch weight and

$$Q : \mathcal{K} \to \mathbb{R}$$

is a fixed nonzero canonical scalar channel on that realized degree–2 channel, then there exists

$$c \in \mathbb{R}$$

such that

$$W = c\,Q.$$

Since admissible branch weights are nonnegative and nontrivial on realizable alternatives, the proportionality constant must satisfy $c > 0$. □

14.7.5 Born law

Theorem 14.7.6 (Born rule)**.** *Let*

$$v_1, \dots, v_n \in \mathcal{K}$$

be first-visible alternatives. Under conditions (S1)–(S4)*, the admissible probability law is*

$$P_i = \frac{Q(v_i)}{\sum_{j=1}^{n} Q(v_j)},$$

where Q is the canonical scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet faithfulness condition.

Proof. By Theorem 14.7.3, probabilities arise from a normalized admissible branch weight W:

$$P_i = \frac{W(v_i)}{\sum_j W(v_j)}.$$

By Theorem 14.7.5,

$$W = c\,Q$$

for some $c > 0$. Substituting gives

$$P_i = \frac{c\,Q(v_i)}{\sum_j c\,Q(v_j)} = \frac{Q(v_i)}{\sum_j Q(v_j)}.$$

□

Remark 14.7.7 (Same source as the cosmological scalar). The scalar channel Q appearing in Theorem 14.7.6 is the same canonical degree–2 scalar channel later isolated at the Einstein boundary on the realized degree–2 channel attached to the stabilized quadratic carrier under the inherited second-jet interface hypothesis. Accordingly, the Born probability law and the cosmological scalar term arise from the same scalar projection on that realized channel.

14.8 Quadratic extension of the scalar channel

Under the inherited second-jet faithfulness condition, the canonical scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier extends to a quadratic form after real linearization. Once that quadratic extension is present, the bilinear source is not an additional datum: it is forced by polarization.

Definition 14.8.1 (Real linearized quadratic carrier)**.** Under the inherited second-jet faithfulness condition, let

$$K$$

denote the realized degree–2 channel attached to the stabilized quadratic carrier. Define its real linearization by

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Z}} \mathbb{R}.$$

Theorem 14.8.2 (Scalar polarization bridge)**.** *Under the inherited second-jet faithfulness condition, let*

$$Q_{\mathbb{R}} : K_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$$

be the canonical quadratic extension of the degree–2 scalar channel, with

$$Q(v) = Q_{\mathbb{R}}(v \otimes 1) \qquad (v \in K).$$

Then

$$B_{\mathbb{R}}(x, y) := \frac{1}{2}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x) - Q_{\mathbb{R}}(y)\big)$$

is the unique symmetric bilinear form on $K_{\mathbb{R}}$ whose diagonal is $Q_{\mathbb{R}}$. Its restriction

$$B(v, w) := B_{\mathbb{R}}(v \otimes 1, w \otimes 1)$$

is symmetric and biadditive on K, and

$$Q(v) = B(v, v) \qquad (v \in K).$$

Proof. The map $Q_{\mathbb{R}}$ is a quadratic form on the real vector space $K_{\mathbb{R}}$. Over a real vector space of characteristic not two, polarization assigns to any quadratic form a symmetric bilinear form by

$$B_{\mathbb{R}}(x, y) = \frac{1}{2}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x) - Q_{\mathbb{R}}(y)\big).$$

The quadratic identity for $Q_{\mathbb{R}}$ gives additivity in each variable and homogeneity over $\mathbb{R}$, so $B_{\mathbb{R}}$ is bilinear. Symmetry is immediate from the displayed formula. Its diagonal is

$$B_{\mathbb{R}}(x, x) = \frac{1}{2}\big(Q_{\mathbb{R}}(2x) - 2Q_{\mathbb{R}}(x)\big) = Q_{\mathbb{R}}(x),$$

because $Q_{\mathbb{R}}(2x) = 4Q_{\mathbb{R}}(x)$. If C is any other symmetric bilinear form with $C(x, x) = Q_{\mathbb{R}}(x)$, then

$$C(x, y) = \frac{1}{2}\big(C(x + y, x + y) - C(x, x) - C(y, y)\big),$$

so $C = B_{\mathbb{R}}$. Thus the bilinear form is unique. Restricting $B_{\mathbb{R}}$ to the embedded integral carrier $K \hookrightarrow K_{\mathbb{R}}$ gives a symmetric biadditive pairing $B : K \times K \to \mathbb{R}$. Finally, for $v \in K$,

$$B(v, v) = B_{\mathbb{R}}(v \otimes 1, v \otimes 1) = Q_{\mathbb{R}}(v \otimes 1) = Q(v).$$

□

Corollary 14.8.3 (Polarization identity). *For the forced scalar-polarization bridge of Theorem 14.8.2, the formula*

$$B_{\mathbb{R}}(x, y) = \frac{1}{2}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x) - Q_{\mathbb{R}}(y)\big)$$

recovers the extended bilinear form. Equivalently, because $B_{\mathbb{R}}$ is symmetric,

$$B_{\mathbb{R}}(x, y) = \frac{1}{4}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x - y)\big).$$

Proof. The first identity is just the expansion formula for the quadratic form attached to a symmetric bilinear form:

$$Q_{\mathbb{R}}(x + y) = Q_{\mathbb{R}}(x) + 2B_{\mathbb{R}}(x, y) + Q_{\mathbb{R}}(y).$$

Solving for $B_{\mathbb{R}}(x, y)$ gives

$$B_{\mathbb{R}}(x, y) = \frac{1}{2}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x) - Q_{\mathbb{R}}(y)\big).$$

For the second identity, apply the first formula to $x + y$ and $x - y$:

$$Q_{\mathbb{R}}(x + y) = Q_{\mathbb{R}}(x) + 2B_{\mathbb{R}}(x, y) + Q_{\mathbb{R}}(y),$$

$$Q_{\mathbb{R}}(x - y) = Q_{\mathbb{R}}(x) - 2B_{\mathbb{R}}(x, y) + Q_{\mathbb{R}}(y).$$

Subtracting yields

$$Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x - y) = 4B_{\mathbb{R}}(x, y),$$

hence

$$B_{\mathbb{R}}(x, y) = \frac{1}{4}\big(Q_{\mathbb{R}}(x + y) - Q_{\mathbb{R}}(x - y)\big).$$

□

Corollary 14.8.4 (Diagonal form). *For the forced scalar-polarization bridge of Theorem 14.8.2, there exists a basis of $K_{\mathbb{R}}$ in which*

$$Q_{\mathbb{R}}(x) = \sum_{i=1}^{r} \lambda_i x_i^2, \qquad \lambda_i \geq 0.$$

Proof. By Theorem 14.8.2, $Q_{\mathbb{R}}$ is represented by a symmetric bilinear form $B_{\mathbb{R}}$ on the finite-dimensional real vector space $K_{\mathbb{R}}$. By the spectral theorem for symmetric bilinear forms over $\mathbb{R}$, there exists a basis in which the representing matrix is diagonal:

$$B_{\mathbb{R}} \sim \operatorname{diag}(\lambda_1, \dots, \lambda_r).$$

Since $Q_{\mathbb{R}}(x) = B_{\mathbb{R}}(x, x)$ is nonnegative on the carrier by construction of the scalar channel, each diagonal entry satisfies

$$\lambda_i \geq 0.$$

Therefore in the corresponding coordinates,

$$Q_{\mathbb{R}}(x) = \sum_{i=1}^{r} \lambda_i x_i^2.$$

□

Remark 14.8.5 (Why no amplitude language is introduced here)*.* The present conclusion is purely real-linear. It yields a diagonal quadratic form on $K_{\mathbb{R}}$, but it does not by itself furnish a canonical complex structure. Accordingly the chapter stops at the diagonal real form above. Any later amplitude interpretation must be introduced separately and proved separately.

14.9 Conclusion

We summarize the logical output of the chapter. First, triangle closure does not alter the quadratic graded layer. The first visible transport carrier therefore remains the degree–2 carrier

$$\mathcal{K} \simeq F^2/F^3,$$

and for a finite-generator triangle-closed model its visible channel count is

$$Q(k) = \binom{k}{2} \asymp k^2.$$

Second, the monotone filtration does not merely permit a normalization; it determines one. Under that canonical normalization, this inherited filtration time is the cumulative count of independent first-visible degree–2 classes, so the cumulative first-visible mass is canonically linear in observer-side filtration time:

$$M(T) \asymp T.$$

Combining this with the quadratic channel law yields

$$k(T) \asymp \sqrt{T}.$$

Third, closure and rectangular factorization force mixed-scale splitting, and that splitting yields coarse multiplicativity of causal diamonds:

$$\mu(D(p, q_T)) \asymp T\, Q(k(T)).$$

Substituting the quadratic channel law and the parabolic relation gives

$$\mu(D(p, q_T)) \asymp T^2.$$

Equivalently, in linear refinement scale

$$L \asymp \sqrt{T},$$

one obtains

$$\mu(D_L) \asymp L^4.$$

Thus the macroscopic diamond dimension is determined to be 4. Fourth, under the inherited second-jet faithfulness condition, the observer-side filtration preorder canonically induces the forward and backward causal cones in the first-variation space attached to the realized degree–2 channel of the stabilized quadratic carrier. Those cones are proper, convex, two-sided, and exhaustive for timelike directions. Accordingly the first-variation pairing extracted, under the inherited second-jet faithfulness condition, from the realized degree–2 channel attached to the stabilized quadratic carrier cannot be definite, cannot have split signature, and therefore, when nondegenerate, must be Lorentzian:

$$\operatorname{sign}(B_p) \in \{(1, 3), (3, 1)\}.$$

Finally, under the inherited second-jet faithfulness condition, the realized degree–2 channel attached to the stabilized quadratic carrier controls the scalar sector. Its admissible scalarizations form a one-dimensional space, and this determines the probability law for first-visible alternatives to be the normalized canonical scalar channel on that realized channel:

$$P_i = \frac{Q(v_i)}{\sum_j Q(v_j)}.$$

After real linearization of that realized degree–2 channel under the inherited second-jet faithfulness condition, the canonical scalar channel extends to a nonnegative quadratic form on

$$K_{\mathbb{R}} = K \otimes_{\mathbb{Z}} \mathbb{R},$$

canonically diagonalizable over $\mathbb{R}$. The chapter therefore establishes a single rigid conclusion:

once the first visible obstruction is quadratic and the inherited observer-side filtration is fixed, the closed comparison stack determines quadratic channel count, parabolic refinement scaling, four-dimensional causal-diamond growth, and Lorentzian first-variation signature.

No additional macroscopic dimension law and no alternative nondegenerate causal signature are compatible with the closed relational architecture developed up to this point once the stabilized quadratic carrier and the inherited observer-side filtration are fixed.

Remark 14.9.1 (Transition to the next chapter). The present chapter has extracted the large-scale causal consequences of the stabilized quadratic carrier together with the inherited observer-side filtration. Chapter 15 follows the parallel linear branch: instead of deriving further causal scaling laws, it reconstructs Hilbert, phase, and scalar structure from the same stabilized quadratic layer. This preserves the geometric and Hilbert realizations as parallel canonical outputs of one intrinsic quadratic obstruction package.

Chapter 15

Complex Hilbert Structure from Quadratic Transport

15.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the Hilbert-reconstruction consequences of the transport-visibility clause item (SP5) on the stabilized quadratic carrier. From the quadratic transport carrier, it derives Hilbert structure and supplies the phase and scalar machinery used in chapters 16 and 18. Theorems 13.10.2 and 13.10.4 identify the stabilized quadratic transport carrier

$$\mathcal{K} \simeq F^2/F^3$$

as the first intrinsic layer at which comparison transport fails to close strictly. This carrier is not an auxiliary linearization, nor an externally imposed state space. It is the first quotient in the intrinsic transport filtration on which nontrivial comparison defect survives. Equivalently, it is the first level of the closed stack at which transport retains information beyond first-order flatness. Two structures have already been determined on this quadratic layer. The first is scalar. The probability law for mutually exclusive first-visible alternatives is tracked through their classes in $\mathcal{K}$, while, under the inherited second-jet faithfulness condition, it is determined by the unique positive quadratic scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier. Thus the first-visible defect sector is not merely a collection of defect classes. Through that realized channel it already carries a distinguished positive quadratic law. The second is orientational. Quadratic defects arise from oriented comparison triangles. Reversal of triangle orientation reverses the corresponding quadratic defect. Hence the same carrier $\mathcal{K}$ carries a canonical reversal symmetry induced by triangle orientation reversal. The purpose of the present chapter is to show that these two facts already force complex Hilbert structure. The subtle point is exact. An involution by itself does not produce a complex structure: an operator squaring to the identity is not an operator squaring to minus the identity. The missing step is that the quadratic transport carrier is not to be treated as an unoriented defect space. Its primitive generators come in two ordered orientation sectors, related by the unique reversal involution. Once that ordered orientation double is made explicit, the scalar form and the reversal symmetry together force a quarter-turn operator, unique up to sign, and hence a complex structure. The residual sign ambiguity is structural rather than arbitrary. It is the necessary symmetry

$$J \mapsto -J,$$

which corresponds to complex conjugation of the same complex Hilbert structure and not to a second geometric choice. Once J is determined, the Hermitian inner product is in turn determined by the quadratic scalar channel and polarization. At the end no further datum is chosen. The resulting classification theorem is the following.

Theorem 15.1.1 (Hilbert structure of the quadratic defect sector)**.** *Under the standing principle Standing Principle 1, let*

$$\mathcal{K} \;\simeq\; F^2/F^3$$

be the stabilized quadratic transport carrier of the closed comparison stack. Then the following structures are determined intrinsically:

(i) *under the inherited second-jet faithfulness condition, a positive definite quadratic scalar form*

$$Q : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}, \qquad \mathcal{K}_{\mathbb{R}} := \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R},$$

extending the canonical degree–2 scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier;

(ii) *a unique filtration-compatible orientation involution*

$$\sigma : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$$

induced by reversal of primitive triangle orientation;

(iii) *a canonical oriented defect double*

$$H_{\mathbb{R}} := \mathcal{K}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O},$$

where $\mathcal{O}$ is the real orientation module generated by the two primitive triangle orientations;

(iv) *an orthogonal complex structure*

$$J : H_{\mathbb{R}} \to H_{\mathbb{R}}, \qquad J^2 = -\mathrm{id},$$

unique up to sign among orthogonal endomorphisms compatible with total orientation reversal and with the quadratic transport filtration;

(v) *a unique Hermitian inner product on $H_{\mathbb{R}}$ whose norm-square extends the quadratic scalar channel.*

Consequently the completion of $H_{\mathbb{R}}$ in the induced norm is a complex Hilbert space canonically associated to the quadratic defect sector, and, under that same inherited second-jet interface hypothesis, the probability law on first-visible alternatives becomes the norm-square law through the realized degree–2 channel attached to the stabilized quadratic carrier.

The proof occupies the remainder of the chapter. The argument is not a free construction. Each stage removes a genuine ambiguity. The quadratic scalar channel determines the real inner product. The orientation reversal is determined uniquely by the primitive transport triangles. The orientation module carries a unique quarter-turn compatible with reversal, up to overall sign. The full oriented defect double then carries a unique orthogonal complex structure anti-commuting with total orientation reversal, again up to that same harmless sign. Finally the Hermitian form is determined uniquely by polarization. At the end nothing remains to be chosen except the sign $J \mapsto -J$, and that sign does not represent additional geometry.

15.2 The quadratic defect sector and its scalar form

Let

$$\mathcal{K} \;\simeq\; F^2/F^3$$

denote the stabilized quadratic transport carrier. Theorems 13.10.2 and 13.13.1 show that $\mathcal{K}$ is generated by the quadratic wedge classes

$$[e_i \wedge e_j],$$

arising from degree–2 commutator residues of oriented comparison transport. We first pass from the intrinsic abelian carrier to its real span.

Definition 15.2.1 (Real quadratic defect space)**.** Define

$$\mathcal{K}_{\mathbb{R}} := \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The scalar-channel theory of Theorems 14.7.3, 14.7.4, 14.7.6 and 14.8.2 yields a canonical positive quadratic law on this real vector space. To use it here, one must formulate the extension carefully. The point is not merely that a scalar law exists on the integral carrier, but that the previously established quadratic-extension principle applies to the stabilized degree–2 transport sector.

Theorem 15.2.2 (Quadratic scalar channel)**.** *Under the inherited second-jet faithfulness condition, there exists a unique positive quadratic form*

$$Q : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$$

extending the canonical degree–2 scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier, whose restriction to first-visible alternatives tracked through the integral carrier $\mathcal{K}$ is the admissible branch-weight law.

Proof. The prior results Theorems 13.10.2, 14.7.3, 14.7.4 and 14.8.2 establish three facts. First, first-visible alternatives are tracked through the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3.$$

Second, under the inherited second-jet faithfulness condition, the admissible branch-weight law on those first-visible alternatives is the canonical degree–2 scalar channel on the realized degree–2 channel attached to that stabilized quadratic carrier, unique up to overall normalization among degree–2 scalarizations. Third, the quadratic-extension theorem applies to that realized-channel scalar law and extends it uniquely to the real linearization $\mathcal{K}_{\mathbb{R}} = \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R}$. Applying that extension theorem in the present inherited-interface regime yields a unique quadratic form

$$Q : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$$

whose restriction to first-visible alternatives tracked through $\mathcal{K}$ is the admissible branch-weight law. Because that law is nonnegative on first-visible alternatives, the extended quadratic form is nonnegative as well. This is the required map Q. □

Remark 15.2.3 (What is used here)*.* Theorem 15.2.2 does not introduce a new scalar law. It imports, under the inherited second-jet faithfulness condition, the canonical degree–2 scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier and extends that law to the real span $\mathcal{K}_{\mathbb{R}}$. Thus the scalar geometry entering the Hilbert construction is already present in the transport analysis of Theorems 13.10.2, 14.7.6 and 14.8.2.

The decisive point is that positivity is strict, not merely weak.

Lemma 15.2.4 (Strict positivity of the quadratic scalar channel)**.** *For every nonzero vector*

$$x \in \mathcal{K}_{\mathbb{R}},$$

one has

$$Q(x) > 0.$$

Equivalently,

$$Q(x) = 0 \Longrightarrow x = 0.$$

Proof. We first prove the claim on the integral carrier $\mathcal{K}$. Let

$$0 \neq k \in \mathcal{K}.$$

Since $\mathcal{K} \simeq F^2/F^3$ is the stabilized first visible transport-defect carrier, a nonzero class k is, by definition, a nontrivial first-visible defect class. The scalar-channel theorem identifies the admissible branch-weight law as the unique positive scalar detector of such first-visible classes. Hence

$$Q(k) > 0 \qquad \text{for every } 0 \neq k \in \mathcal{K}.$$

Now let

$$0 \neq x \in \mathcal{K}_{\mathbb{R}}.$$

Choose an integral basis

$$k_1, \ldots, k_m$$

of the free part of $\mathcal{K}$, so that

$$x = \sum_{a=1}^{m} \lambda_a (k_a \otimes 1)$$

with real coefficients λ_a, not all zero. By density of rational coefficients and homogeneity of degree 2, it suffices to prove strict positivity on nonzero rational combinations. Thus choose $N > 0$ such that

$$Nx \in \mathcal{K}$$

and $Nx \neq 0$. Then, by the integral case already proved,

$$Q(Nx) > 0.$$

Since Q is quadratic,

$$Q(Nx) = N^2 Q(x).$$

Therefore

$$Q(x) = N^{-2} Q(Nx) > 0.$$

This proves strict positivity on all nonzero vectors of $\mathcal{K}_{\mathbb{R}}$. □

The quadratic form determines a symmetric bilinear form by polarization.

Definition 15.2.5 (Real polarization)**.** Define

$$g_{\mathcal{K}}(x, y) := \frac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big) \qquad (x, y \in \mathcal{K}_{\mathbb{R}}).$$

Proposition 15.2.6 (Real inner product on the quadratic carrier)**.** *The form $g_{\mathcal{K}}$ is a positive definite symmetric bilinear form on $\mathcal{K}_{\mathbb{R}}$.*

Proof. Because Q is a quadratic form on the real vector space $\mathcal{K}_{\mathbb{R}}$, there exists a unique symmetric bilinear form whose diagonal is Q, and the displayed formula is exactly its polarization. Thus $g_{\mathcal{K}}$ is symmetric and bilinear. Moreover, for every $x \in \mathcal{K}_{\mathbb{R}}$,

$$g_{\mathcal{K}}(x, x) = Q(x) \geq 0.$$

If

$$g_{\mathcal{K}}(x, x) = 0,$$

then

$$Q(x) = 0,$$

and Theorem 15.2.4 implies

$$x = 0.$$

Therefore $g_{\mathcal{K}}$ is positive definite. □

Remark 15.2.7 (No complex structure yet). At this stage the quadratic carrier has acquired a canonical real Euclidean geometry and nothing more. The scalar form determines $g_{\mathcal{K}}$, but a real inner product alone does not force complex structure. The next task is to isolate the missing orientational input.

15.3 Orientation reversal on the quadratic carrier

The next step is to isolate the orientational structure determined by comparison triangles.

Lemma 15.3.1 (Orientation involution). *Primitive triangle orientation reversal induces a linear involution*

$$\sigma : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$$

satisfying

$$\sigma^2 = \mathrm{id}, \qquad \sigma([e_i \wedge e_j]) = -[e_i \wedge e_j].$$

Proof. Let (i, j, ℓ) be an oriented primitive comparison triangle. Reversal of orientation sends (i, j, ℓ) to (ℓ, j, i). At the level of transport words, this reverses the traversal order of the corresponding elementary comparison steps. Passing to the quadratic quotient

$$F^2/F^3,$$

reversal of order changes the sign of the associated wedge class. Thus

$$[e_i \wedge e_j] \longmapsto -[e_i \wedge e_j].$$

Because primitive triangle reversal is itself an involution, the induced map on $\mathcal{K}$, and hence on $\mathcal{K}_{\mathbb{R}}$, squares to the identity. Linearity is immediate from passage to the real span. □

Lemma 15.3.2 (Orthogonality of the orientation involution). *The involution σ preserves the quadratic scalar form and is orthogonal with respect to $g_{\mathcal{K}}$:*

$$Q(\sigma x) = Q(x), \qquad g_{\mathcal{K}}(\sigma x, \sigma y) = g_{\mathcal{K}}(x, y).$$

Proof. Orientation reversal changes a primitive quadratic defect to its negative. Since the scalar channel is quadratic, it is invariant under sign:

$$Q(-x) = Q(x) \qquad (x \in \mathcal{K}_{\mathbb{R}}).$$

By Theorem 15.3.1, σ acts on primitive generators by sign reversal, hence

$$Q(\sigma x) = Q(x)$$

on the spanning set of primitive generators, and therefore on all of $\mathcal{K}_{\mathbb{R}}$ by quadraticity and linear extension. Orthogonality of σ with respect to $g_{\mathcal{K}}$ now follows by polarization:

$$g_{\mathcal{K}}(\sigma x, \sigma y) = \frac{1}{2}\big(Q(\sigma x + \sigma y) - Q(\sigma x) - Q(\sigma y)\big) = \frac{1}{2}\big(Q(\sigma(x+y)) - Q(x) - Q(y)\big) = g_{\mathcal{K}}(x, y).$$

□

At this point one has an involution, but not yet uniqueness. That distinction matters. The later complex structure must not depend on a discretionary choice of reversal operator. One must show that the transport filtration itself determines the orientation involution and excludes all competitors.

Lemma 15.3.3 (Uniqueness of the orientation involution)**.** *Let*

$$\tau : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$$

be a linear automorphism satisfying the following properties:

(i) τ *is induced by an automorphism of the primitive triangle transport system preserving the quadratic filtration and passage to* F^2/F^3*;*

(ii) τ *reverses primitive triangle orientation;*

(iii) $\tau^2 = \mathrm{id}$.

Then

$$\tau = \sigma.$$

Proof. The quadratic carrier $\mathcal{K}_{\mathbb{R}}$ is spanned by the wedge classes

$$[e_i \wedge e_j].$$

Because τ is filtration-compatible, its action on the entire carrier is determined by its action on this generating family. Now τ reverses primitive triangle orientation. Therefore on each primitive wedge generator it acts exactly as orientation reversal acts:

$$\tau([e_i \wedge e_j]) = -[e_i \wedge e_j].$$

By Theorem 15.3.1, the same formula holds for σ. Hence τ and σ agree on a spanning set of $\mathcal{K}_{\mathbb{R}}$, and therefore agree everywhere. ☐

Remark 15.3.4 (Rigidity of reversal). The involution σ is therefore not merely available. It is the unique filtration-compatible realization of triangle orientation reversal on the quadratic carrier. This rigidity is what prevents the later complex structure from hiding a new arbitrary choice.

15.3.1 Uniqueness of orientation splitting on the quadratic carrier

The previous lemmas can now be assembled into the object-locus uniqueness statement used later in chapter 19.

Theorem 15.3.5 (Uniqueness of orientation splitting)**.** *Under the standing principle Standing Principle 1, there exists a nontrivial orientation involution*

$$R : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}, \qquad R^2 = \mathrm{id},$$

reversing primitive oriented quadratic observables. Moreover, up to conjugacy by admissible filtration-compatible automorphisms of $\mathcal{K}_{\mathbb{R}}$, this involution is unique. Its nontrivial action determines a canonical minimal irreducible real block on which there is an orthogonal quarter-turn operator J satisfying

$$J^2 = -\,\mathrm{id}, \qquad JR = -RJ.$$

Proof. **Step 1 (transport visibility: item (SP5)).** Existence of a nontrivial orientation involution is Theorem 15.3.1. Orthogonality and filtration compatibility are Theorem 15.3.2.

Step 2 (intrinsic and quotient admissibility: items (SP1) and (SP4)). Uniqueness among admissible orientation-reversing involutions is Theorem 15.3.3. Thus the orientation reversal channel is fixed up to admissible filtration-compatible equivalence.

Step 3 (intrinsic minimal block extraction: item (SP1)). Let $V \subseteq \mathcal{K}_{\mathbb{R}}$ be a minimal nontrivial R-stable real subspace. Pick $0 \neq v \in V$. Since R is nontrivial, Rv is not proportional to v, and minimality gives

$$V = \mathrm{span}_{\mathbb{R}}\{v, Rv\}.$$

Define

$$Jv := Rv, \qquad J(Rv) := -v,$$

and extend linearly. Then

$$J^2 = -\,\mathrm{id}, \qquad JR = -RJ.$$

Because R is orthogonal and V is generated by an R-orbit pair, J is orthogonal on V. Conjugacy uniqueness follows from uniqueness of the involution and filtration-compatible transport equivalence. ☐

Corollary 15.3.6 (Primitive doublet)**.** *The minimal irreducible real representation carrying the pair (R, J) of Theorem 15.3.5 is two-dimensional. Hence orientation splitting determines a primitive multiplicity generator of dimension* 2*.*

Proof. On the minimal nontrivial block, R has eigenvalues ± 1 and J exchanges the eigendirections because $JR = -RJ$. Thus at least two real dimensions are required. Minimality of the block excludes larger dimension. ☐

15.4 The orientation module

An involution on $\mathcal{K}_{\mathbb{R}}$ does not by itself produce a complex structure. The relevant object is not the bare carrier $\mathcal{K}_{\mathbb{R}}$, but the ordered double of its two primitive orientation sectors. We now make that double explicit.

Definition 15.4.1 (Orientation module)**.** Let $\mathcal{O}$ be the two-dimensional real vector space with basis

$$\varepsilon_+,\ \varepsilon_-,$$

representing the two primitive triangle orientations. Equip $\mathcal{O}$ with the Euclidean inner product $h_{\mathcal{O}}$ for which this basis is orthonormal, and define the reversal involution

$$r : \mathcal{O} \to \mathcal{O}$$

by

$$r(\varepsilon_+) = \varepsilon_-, \qquad r(\varepsilon_-) = \varepsilon_+.$$

The key point is that $\mathcal{O}$ carries a quarter-turn compatible with reversal, and that this quarter-turn is unique up to sign.

Proposition 15.4.2 (Quarter-turn on the orientation module)**.** *There exists an orthogonal endomorphism*

$$j_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$$

such that

$$j_{\mathcal{O}}^2 = -\mathrm{id}, \qquad j_{\mathcal{O}} r = -r j_{\mathcal{O}}.$$

It is unique up to overall sign. Explicitly,

$$j_{\mathcal{O}}(\varepsilon_+) = \varepsilon_-, \qquad j_{\mathcal{O}}(\varepsilon_-) = -\varepsilon_+.$$

Proof. Define $j_{\mathcal{O}}$ by the displayed formulas. Then

$$j_{\mathcal{O}}^2(\varepsilon_+) = j_{\mathcal{O}}(\varepsilon_-) = -\varepsilon_+, \qquad j_{\mathcal{O}}^2(\varepsilon_-) = j_{\mathcal{O}}(-\varepsilon_+) = -\varepsilon_-,$$

so

$$j_{\mathcal{O}}^2 = -\mathrm{id}.$$

In the ordered orthonormal basis $\{\varepsilon_+, \varepsilon_-\}$, the matrix of $j_{\mathcal{O}}$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

hence $j_{\mathcal{O}}$ is orthogonal. Next,

$$j_{\mathcal{O}} r(\varepsilon_+) = j_{\mathcal{O}}(\varepsilon_-) = -\varepsilon_+, \qquad -r j_{\mathcal{O}}(\varepsilon_+) = -r(\varepsilon_-) = -\varepsilon_+,$$

and similarly on ε_-, so

$$j_{\mathcal{O}} r = -r j_{\mathcal{O}}.$$

For uniqueness, let $j'_{\mathcal{O}}$ be another orthogonal endomorphism satisfying

$$(j'_{\mathcal{O}})^2 = -\mathrm{id}, \qquad j'_{\mathcal{O}} r = -r j'_{\mathcal{O}}.$$

The operator r has eigenspaces spanned by

$$\varepsilon_+ + \varepsilon_- \qquad \text{and} \qquad \varepsilon_+ - \varepsilon_-.$$

The anti-commutation relation determines $j'_{\mathcal{O}}$ to exchange these two lines. Orthogonality and the identity $(j'_{\mathcal{O}})^2 = -\mathrm{id}$ then leave exactly two possibilities:

$$j'_{\mathcal{O}} = j_{\mathcal{O}} \qquad \text{or} \qquad j'_{\mathcal{O}} = -j_{\mathcal{O}}.$$

□

Remark 15.4.3 (Meaning of the sign ambiguity). The sign ambiguity in $j_{\mathcal{O}}$ is necessary and harmless. It is the ordinary ambiguity between a complex structure and its conjugate. It does not encode a second orientation geometry on the closed stack.

15.5 The oriented defect double and its universal property

We now form the real vector space on which the complex structure will live.

Definition 15.5.1 (Oriented defect double)**.** Define

$$H_{\mathbb{R}} := \mathcal{K}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}.$$

Equip $H_{\mathbb{R}}$ with the symmetric bilinear form

$$g(x \otimes u,\, y \otimes v) := g_{\mathcal{K}}(x, y)\, h_{\mathcal{O}}(u, v),$$

extended bilinearly to all of $H_{\mathbb{R}}$.

Proposition 15.5.2 (Product metric)**.** *The form g is a positive definite symmetric bilinear form on $H_{\mathbb{R}}$.*

Proof. Symmetry and bilinearity are immediate from symmetry and bilinearity of $g_{\mathcal{K}}$ and $h_{\mathcal{O}}$. It remains to prove positive definiteness. Let

$$\xi = \sum_{a=1}^{m} x_a \otimes u_a \in H_{\mathbb{R}}.$$

Choose the orthonormal basis

$$\varepsilon_+, \varepsilon_-$$

of $\mathcal{O}$ from Theorem 15.4.1. Then ξ may be written uniquely in the form

$$\xi = x_+ \otimes \varepsilon_+ + x_- \otimes \varepsilon_-$$

for some $x_+, x_- \in \mathcal{K}_\mathbb{R}$. Since the basis is orthonormal, one has

$$g(\xi, \xi) = g_\mathcal{K}(x_+, x_+) + g_\mathcal{K}(x_-, x_-).$$

By Theorem 15.2.6, the form $g_\mathcal{K}$ is positive definite. Hence

$$g(\xi, \xi) \geq 0,$$

with equality if and only if

$$x_+ = 0 \qquad \text{and} \qquad x_- = 0.$$

Thus $\xi = 0$. Therefore g is positive definite. □

The two sources of involution on $H_\mathbb{R}$ are the quadratic orientation involution σ on $\mathcal{K}_\mathbb{R}$ and the sector-reversal involution r on $\mathcal{O}$. Together they determine the global orientation-reversal symmetry.

Definition 15.5.3 (Total orientation reversal)**.** Define

$$\Sigma := \sigma \otimes r : H_\mathbb{R} \to H_\mathbb{R}.$$

Proposition 15.5.4 (Total reversal is orthogonal)**.** *The operator* Σ *is an orthogonal involution:*

$$\Sigma^2 = \mathrm{id}, \qquad g(\Sigma\xi, \Sigma\eta) = g(\xi, \eta) \qquad (\xi, \eta \in H_\mathbb{R}).$$

Proof. Since

$$\sigma^2 = \mathrm{id}_{\mathcal{K}_\mathbb{R}} \qquad \text{and} \qquad r^2 = \mathrm{id}_\mathcal{O},$$

one has

$$\Sigma^2 = (\sigma \otimes r)(\sigma \otimes r) = \sigma^2 \otimes r^2 = \mathrm{id}_{H_\mathbb{R}}.$$

To prove orthogonality, it suffices to check the formula on pure tensors. For $x, y \in \mathcal{K}_\mathbb{R}$ and $u, v \in \mathcal{O}$,

$$g(\Sigma(x \otimes u), \Sigma(y \otimes v)) = g(\sigma x \otimes ru, \sigma y \otimes rv) = g_\mathcal{K}(\sigma x, \sigma y)\, h_\mathcal{O}(ru, rv).$$

By Theorem 15.3.2,

$$g_\mathcal{K}(\sigma x, \sigma y) = g_\mathcal{K}(x, y),$$

and by construction of r,

$$h_\mathcal{O}(ru, rv) = h_\mathcal{O}(u, v).$$

Therefore

$$g(\Sigma(x \otimes u), \Sigma(y \otimes v)) = g_\mathcal{K}(x, y)\, h_\mathcal{O}(u, v) = g(x \otimes u, y \otimes v).$$

By bilinearity this holds for all $\xi, \eta \in H_\mathbb{R}$. □

The appropriate quarter-turn on $H_{\mathbb{R}}$ is now determined by the orientation-module quarter-turn. The only remaining issue is to exclude hidden carrier automorphisms.

Lemma 15.5.5 (Rigidity of filtration-compatible carrier automorphisms)**.** *Let*

$$A : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$$

be an orthogonal automorphism satisfying the following properties:

(i) *A is induced functorially from the quadratic transport filtration;*

(ii) *A commutes with the orientation involution:*

$$A\sigma = \sigma A;$$

(iii) *A preserves the primitive wedge generator system up to filtration equivalence.*

Then

$$A = \pm \mathrm{id}_{\mathcal{K}_{\mathbb{R}}}.$$

If in addition A preserves primitive orientation sector by sector, then

$$A = \mathrm{id}_{\mathcal{K}_{\mathbb{R}}}.$$

Proof. The carrier $\mathcal{K}_{\mathbb{R}}$ is generated by the primitive quadratic wedge classes

$$[e_i \wedge e_j].$$

By filtration-functoriality and preservation of the primitive generator system up to filtration equivalence, the action of A is determined by its action on these generators. Since A commutes with σ and σ acts on each primitive generator by sign reversal,

$$\sigma([e_i \wedge e_j]) = -[e_i \wedge e_j],$$

the image under A of a primitive generator must again lie in the same one-dimensional orientation line generated by $[e_i \wedge e_j]$. Indeed, if A mixed distinct primitive lines, then commuting with the unique orientation involution would produce an additional filtration-compatible intrinsic symmetry on the primitive quadratic transport carrier beyond the determined reversal symmetry, contradicting the rigidity of primitive transport classification in Theorems 15.1.1 and 15.3.3. Therefore, for each primitive generator,

$$A([e_i \wedge e_j]) = \lambda_{ij}[e_i \wedge e_j]$$

for some real scalar λ_{ij}. Because A is orthogonal with respect to the positive definite form $g_{\mathcal{K}}$, each λ_{ij} satisfies

$$\lambda_{ij}^2 = 1,$$

hence

$$\lambda_{ij} = \pm 1.$$

It remains to show that the sign is global rather than generatorwise. If different primitive generators carried different signs, then A would separate distinct primitive transport directions by an additional filtration-compatible orthogonal datum not already encoded by the quadratic transport system. But the quadratic rigidity results (Theorems 15.1.1 and 15.3.3) identify the primitive degree–2 carrier only through the scalar form and the unique orientation involution. Accordingly no further generatorwise sign freedom exists. Hence all λ_{ij} coincide, and therefore

$$A = \pm \mathrm{id}_{\mathcal{K}_{\mathbb{R}}}.$$

If A preserves primitive orientation sector by sector, then the global negative sign is excluded, and one obtains

$$A = \mathrm{id}_{\mathcal{K}_{\mathbb{R}}}.$$

□

Theorem 15.5.6 (Universal complex structure on the oriented defect double)**.** *There exists an orthogonal endomorphism*

$$J : H_{\mathbb{R}} \to H_{\mathbb{R}}$$

such that

$$J^2 = -\mathrm{id}, \qquad J\Sigma = -\Sigma J.$$

It is unique up to overall sign among orthogonal endomorphisms with these properties and compatible with the quadratic transport filtration. Explicitly,

$$J := \mathrm{id}_{\mathcal{K}_{\mathbb{R}}} \otimes j_{\mathcal{O}}.$$

Proof. Define

$$J := \mathrm{id}_{\mathcal{K}_{\mathbb{R}}} \otimes j_{\mathcal{O}}.$$

Then

$$J^2 = \mathrm{id}_{\mathcal{K}_{\mathbb{R}}} \otimes j_{\mathcal{O}}^2 = -\mathrm{id}_{H_{\mathbb{R}}},$$

because $j_{\mathcal{O}}^2 = -\mathrm{id}_{\mathcal{O}}$ by Theorem 15.4.2. Orthogonality follows immediately from orthogonality of $j_{\mathcal{O}}$ and orthogonality of the identity on $\mathcal{K}_{\mathbb{R}}$. Next,

$$J\Sigma = (\mathrm{id} \otimes j_{\mathcal{O}})(\sigma \otimes r) = \sigma \otimes (j_{\mathcal{O}} r) = -\sigma \otimes (r j_{\mathcal{O}}) = -\Sigma J,$$

because $j_{\mathcal{O}} r = -r j_{\mathcal{O}}$. This proves existence. We now prove uniqueness up to sign. Let

$$J' : H_{\mathbb{R}} \to H_{\mathbb{R}}$$

be an orthogonal endomorphism satisfying

$$(J')^2 = -\mathrm{id}, \qquad J'\Sigma = -\Sigma J',$$

and assume J' is compatible with the quadratic transport filtration. Because J' is filtration-compatible, its action on the $\mathcal{K}_{\mathbb{R}}$-factor is induced from a filtration-compatible orthogonal automorphism

$$A : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$$

commuting with the unique orientation involution σ. By Theorem 15.5.5, such an A equals

$$A = \pm \mathrm{id}_{\mathcal{K}_{\mathbb{R}}}.$$

Absorbing this harmless global sign into the orientation factor, one may therefore write

$$J' = \mathrm{id}_{\mathcal{K}_{\mathbb{R}}} \otimes j'_{\mathcal{O}}$$

for some orthogonal endomorphism

$$j'_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}.$$

The identities for J' now become

$$(j'_{\mathcal{O}})^2 = -\mathrm{id}, \qquad j'_{\mathcal{O}} r = -r j'_{\mathcal{O}}.$$

By Theorem 15.4.2, the only orthogonal endomorphisms of $\mathcal{O}$ satisfying these identities are

$$j'_{\mathcal{O}} = j_{\mathcal{O}} \qquad \text{or} \qquad j'_{\mathcal{O}} = -j_{\mathcal{O}}.$$

Therefore

$$J' = \pm J.$$

This proves uniqueness up to sign. □

Remark 15.5.7 (Where complex structure first appears). Theorem 15.5.6 is the precise point at which complex structure becomes necessary. The scalar geometry on $\mathcal{K}_{\mathbb{R}}$ supplies the real metric. The ordered orientation double supplies the quarter-turn. The compatibility condition with total orientation reversal removes all remaining freedom except the necessary sign $J \mapsto -J$.

Proposition 15.5.8 (Uniqueness of the orthogonal complex structure)**.** *Let*

$$H_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}$$

be the oriented defect double, equipped with the product inner product g, and let

$$\Sigma = \sigma \otimes r$$

be the total orientation-reversal involution. Suppose

$$J' : H_{\mathbb{R}} \to H_{\mathbb{R}}$$

is an orthogonal endomorphism satisfying

$$(J')^2 = -\mathrm{id}, \qquad J'\Sigma = -\Sigma J',$$

and is induced functorially from the quadratic transport carrier. Then

$$J' = \pm J,$$

where

$$J = \mathrm{id}_{\mathcal{K}_{\mathbb{R}}} \otimes j_{\mathcal{O}}$$

is the canonical complex structure from Theorem 15.5.6.

Proof. This is the uniqueness statement already proved in Theorem 15.5.6, stated separately for later reference. The filtration-compatible functoriality condition reduces the carrier part to the identity up to harmless global sign by Theorem 15.5.5, and the orientation quarter-turn is unique up to sign by Theorem 15.4.2. Therefore $J' = \pm J$. □

In particular, once filtration compatibility and anti-commutation with total orientation reversal are imposed, the complex structure on the oriented defect double is determined uniquely up to the harmless sign ambiguity

$$J \mapsto -J.$$

This sign is structural. It does not encode a second geometry; it is the analogue of passing from a complex structure to its conjugate.

15.6 Skew form and Hermitian structure

The canonical complex structure J determines a corresponding skew form.

Definition 15.6.1 (Canonical skew form)**.** Define

$$\omega(\xi, \eta) := g(\xi, J\eta) \qquad (\xi, \eta \in H_{\mathbb{R}}).$$

Proposition 15.6.2 (Properties of the skew form)**.** *The form ω is bilinear, antisymmetric, and nondegenerate.*

Proof. Bilinearity is immediate from bilinearity of g and linearity of J. To prove antisymmetry, we first note that J is skew-adjoint with respect to g. Indeed, orthogonality of J and the identity $J^2 = -\mathrm{id}$ give, for all $\xi, \eta \in H_{\mathbb{R}}$,

$$g(J\xi, \eta) = g(J^2\xi, J\eta) = -g(\xi, J\eta).$$

Hence

$$\omega(\eta, \xi) = g(\eta, J\xi) = -g(J\eta, \xi) = -g(\xi, J\eta) = -\omega(\xi, \eta).$$

To prove nondegeneracy, suppose $\omega(\xi, \eta) = 0$ for all $\eta \in H_{\mathbb{R}}$. Then

$$g(\xi, J\eta) = 0 \qquad \text{for all } \eta.$$

Since J is invertible, every vector $\zeta \in H_{\mathbb{R}}$ may be written as $\zeta = J\eta$ for some η. Therefore

$$g(\xi, \zeta) = 0 \qquad \text{for all } \zeta \in H_{\mathbb{R}}.$$

Because g is nondegenerate, this determines $\xi = 0$. Thus ω is nondegenerate. $\square$

We now isolate the rigidity statement that removes the last remaining loophole in the passage from the real metric to the Hermitian form.

Lemma 15.6.3 (Polarization rigidity)**.** *Let $H_{\mathbb{R}}$ be the real oriented defect sector, let*

$$Q_H(\xi) := g(\xi, \xi)$$

be the quadratic form induced by g, and let

$$J : H_{\mathbb{R}} \to H_{\mathbb{R}}$$

be the canonical complex structure of Theorem 15.5.6. Then there exists a unique Hermitian inner product

$$\langle \cdot, \cdot \rangle : H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{C}$$

whose real part is g, whose imaginary part is compatible with J, and whose norm-square satisfies

$$\langle \xi, \xi \rangle = Q_H(\xi) \qquad (\xi \in H_{\mathbb{R}}).$$

More explicitly, the unique such Hermitian form is

$$\langle \xi, \eta \rangle = g(\xi, \eta) + i\, g(\xi, J\eta).$$

Proof. We first prove uniqueness. Let

$$\langle \cdot, \cdot \rangle$$

be any Hermitian form satisfying the stated properties. Its real part is determined uniquely by the norm-square Q_H, because real polarization gives

$$\Re\langle \xi, \eta \rangle = \frac{1}{2}\big(Q_H(\xi + \eta) - Q_H(\xi) - Q_H(\eta)\big) = g(\xi, \eta).$$

Thus the real part is determined. The compatibility with J determines the imaginary part as well. Indeed, by definition of compatibility, the imaginary part must be the skew form determined by g and J, namely

$$\Im\langle \xi, \eta \rangle = g(\xi, J\eta).$$

Hence the Hermitian form is uniquely determined and must equal

$$g(\xi, \eta) + i\, g(\xi, J\eta).$$

It remains to verify that this displayed formula defines a Hermitian inner product. Conjugate symmetry follows from symmetry of g and skew-adjointness of J:

$$\overline{\langle \xi, \eta \rangle} = g(\xi, \eta) - i\, g(\xi, J\eta) = g(\eta, \xi) + i\, g(\eta, J\xi) = \langle \eta, \xi \rangle.$$

Real bilinearity of g and linearity of J imply complex sesquilinearity in the usual way when $H_{\mathbb{R}}$ is regarded as a complex vector space via J. Finally, for every $\xi \in H_{\mathbb{R}}$,

$$\langle \xi, \xi \rangle = g(\xi, \xi) + i\, g(\xi, J\xi).$$

But antisymmetry of the skew form gives

$$g(\xi, J\xi) = \omega(\xi, \xi) = 0.$$

Therefore

$$\langle \xi, \xi \rangle = g(\xi, \xi) = Q_H(\xi) \geq 0.$$

If $\langle \xi, \xi \rangle = 0$, then $Q_H(\xi) = 0$, and since g is positive definite this determines $\xi = 0$. Thus the form is positive definite. Hence the displayed formula defines the unique Hermitian inner product with the required properties. □

Remark 15.6.4 (No residual freedom). Once the quadratic scalar law and the canonical complex structure are fixed, no further freedom remains. The Hermitian inner product is not chosen. It is rigidly determined by the real metric and the quarter-turn through polarization.

15.7 Hilbert reconstruction

We may now formulate the Hilbert reconstruction theorem in its final form.

Theorem 15.7.1 (Hilbert reconstruction)**.** *The quadratic scalar channel Q on the quadratic carrier and the canonical complex structure J on the oriented defect double determine a unique Hermitian inner product on $H_{\mathbb{R}}$, namely the form of Theorem 15.6.3. The completion of $H_{\mathbb{R}}$ in the associated norm is a complex Hilbert space.*

Proof. By Theorem 15.6.3, there is a unique Hermitian inner product on $H_{\mathbb{R}}$ whose norm-square is the induced quadratic form

$$Q_H(\xi) = g(\xi, \xi)$$

and whose imaginary part is determined by the canonical complex structure J. Thus $H_{\mathbb{R}}$, regarded as a complex vector space via J, is a pre-Hilbert space. Completing $H_{\mathbb{R}}$ in the norm induced by this Hermitian form produces a complex Hilbert space. □

We denote this completion by H.

15.8 Born rule

The probabilistic interpretation is now immediate.

Corollary 15.8.1 (Born rule)**.** *For every $\xi \in H$,*

$$P(\xi) = \|\xi\|^2.$$

In particular, the probabilistic scalar channel of the quadratic defect sector is exactly the squared norm of the uniquely determined Hermitian inner product.

Proof. By construction of the Hermitian inner product,

$$\|\xi\|^2 = \langle \xi, \xi \rangle = Q_H(\xi).$$

But Q_H is precisely the quadratic scalar channel induced from the unique probability law on first-visible alternatives. Therefore

$$P(\xi) = \|\xi\|^2.$$

□

Remark 15.8.2 (Meaning of the norm-square law)*.* The Born rule is not postulated here as an external axiom of a Hilbert space formalism. Rather, the Hilbert norm is reconstructed precisely so that its norm-square coincides with the already determined scalar channel on the quadratic defect sector. Thus the norm-square law is a theorem of the transport structure, not an additional probabilistic prescription.

15.9 Interpretation

The logic of the chapter may now be summarized without omission. The closed comparison stack first determines the quadratic transport carrier

$$\mathcal{K} \simeq F^2/F^3.$$

The scalar-channel theorem equips that carrier with a positive definite quadratic law. Triangle reversal equips it with a unique filtration-compatible involution. The ordered orientation double then carries a unique orthogonal complex structure anti-commuting with total orientation reversal, up to the harmless sign ambiguity

$$J \mapsto -J.$$

The quadratic scalar law and this quarter-turn determine, by polarization rigidity, a unique Hermitian inner product. Completion produces a complex Hilbert space. Thus the Hilbert framework is not appended externally to the closed comparison stack. It is the canonical linear realization of the same intrinsic quadratic defect layer whose geometric branch, under the inherited second-jet faithfulness condition, is read on the realized degree–2 channel attached to the stabilized quadratic carrier, where nonzero stabilized square classes are equivalent to nonzero realized curvature. The chain is

$$\begin{aligned}&\text{transport} \implies F^2/F^3 \implies \text{scalar law} \implies \text{orientation double} \\ &\implies J^2 = -\mathrm{id} \implies \text{Hermitian rigidity} \implies \text{Hilbert space.}\end{aligned}$$

The next chapter returns to the parallel geometric branch of the same intrinsic quadratic defect layer. Under the inherited second-jet faithfulness condition, that branch is read on the realized degree–2 channel attached to the stabilized quadratic carrier, where nonzero stabilized square classes are equivalent to nonzero realized curvature. Higher transport jets then begin to contribute genuinely new geometric information beyond that inherited interface bridge.

Remark 15.9.1 (Parallel realizations of the same carrier). The significance of the chapter is not merely that a Hilbert space has been reconstructed. It is that the Hilbert realization is an intrinsic linear realization of the stabilized quadratic defect layer, while the later geometric branch is read, under the inherited second-jet faithfulness condition, on the realized degree–2 channel attached to the stabilized quadratic carrier. The subsequent development therefore does not juxtapose two unrelated frameworks. It develops two linked realizations of one closed comparison structure, with the geometric side entering only through that inherited interface bridge.

15.10 Conclusion

The chapter proves that Hilbert structure is not auxiliary input: the Hermitian form, norm-square probability law, and phase sector are all forced by the stabilized quadratic transport carrier F^2/F^3. Hilbert realization is therefore an intrinsic determination of the closed comparison structure, while the later geometric branch is read, under the inherited second-jet faithfulness condition, on the realized degree–2 channel attached to the stabilized quadratic carrier. Accordingly, chapter 16 continues from that inherited interface bridge to the macroscopic compatibility equation carried by the realized geometric branch.

Chapter 16

Transport Closure and the Uniqueness of the Macroscopic Compatibility Law

16.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the macroscopic compatibility consequences of the repaired intrinsic transport package already established in chapters 13 to 15. Using the stabilized quadratic carrier together with the inherited observer-side filtration and, under the inherited second-jet faithfulness condition, the realized degree–2 channel and first-variation pairing carried there, it derives the unique macroscopic compatibility law used in chapters 17 and 18. Chapters 13 to 15 therefore supply the exact intrinsic transport, filtration, and realized-branch input needed here, rather than a bare whole-carrier curvature identification or an unconditional direct smooth-realization package. At the intrinsic level, the first nontrivial stabilized obstruction is the quadratic transport carrier

$$F^2/F^3$$

on the Stone inverse limit. At the realized level, two structural facts have already been established. First, chapter 13 shows that, under the inherited second-jet faithfulness condition, nonzero stabilized square classes are equivalent to nonzero realized curvature in the realized metric geometry. Second, the pointwise first-variation geometry constructed in Proposition 17.2.2, together with the smooth realization theorem for connection-first geometry in Theorem 17.9.1(ii) and the tangent identification stated in Section 17.12, identifies the intrinsic first-variation arenas with the realized tangent geometry. More precisely, for each realized point p,

$$V_p \cong T_pM,$$

and, under the inherited second-jet faithfulness condition, the first-variation pairing on the realized degree–2 channel attached to the stabilized quadratic carrier

$$B_p : V_p \times V_p \to \mathbb{R},$$

is identified with the Lorentz metric pairing

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

Thus the realized transport geometry contains not only the curvature channel of the quadratic carrier, but also, under that inherited second-jet faithfulness condition, the pointwise quadratic pairing carried by the realized degree–2 channel attached to the stabilized quadratic carrier. In particular, comparison automorphisms preserve the realized transport geometry at the metric level and not merely at the curvature level. The purpose of the present chapter is to determine which macroscopic compatibility laws are admissible for that realized metric geometry. No action functional is assumed. No matter tensor is postulated. No variational principle, entropy extremization principle, or external geometric axiom is inserted. The argument is entirely internal to the closed stack and uses only structures already determined by Theorems 13.13.1, 14.6.9, 15.2.2 and 15.5.6. We work strictly after the development in chapters 13 to 15. In particular, we take as input the following previously established facts.

(S1) Rectangular completeness and canonical quotient semantics.

(S2) Closed-system descent and exhaustion of enrichment by the same two loci, possibly with both and with no third: representative choice and transport.

(S3) Transport closure: every admissible microscopic dynamics lies in

$$\mathsf{G} := \mathrm{Aut}(U, \mathcal{C}).$$

(S4) A refinement tower

$$\mathsf{B}_1 \subset \mathsf{B}_2 \subset \cdots, \qquad \mathsf{B}_\infty = \Big\langle \bigcup_k \mathsf{B}_k \Big\rangle,$$

with Stone inverse limit

$$S_\infty = \mathrm{UF}(\mathsf{B}_\infty) \cong \varprojlim_k \mathrm{UF}(\mathsf{B}_k).$$

(S5) Foundation's quantifier boundary: global admissibility is equivalent to finite non-coverage of the excluded finite comparison cells, equivalently to existence of a coherent admissible ultrafilter tower.

(S6) The stabilized intrinsic degree–2 transport carrier

$$\mathcal{K}_\infty \subset F^2/F^3$$

on the inverse-limit locus.

(S7) Interface realization: under the inherited second-jet faithfulness condition and smooth realization

$$\iota : S_\infty \to M,$$

nonzero stabilized square classes are equivalent to nonzero realized curvature for a smooth realized metric g.

(S8) First-variation realization: by Proposition 17.2.2, Theorem 17.9.1(ii), and Section 17.12, the pointwise first-variation space V_p realizes as the tangent space T_pM, and, under the inherited second-jet faithfulness condition, the first-variation pairing on the realized degree–2 channel attached to the stabilized quadratic carrier

$$B_p : V_p \times V_p \to \mathbb{R}$$

is identified with the Lorentz metric pairing

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

(S9) Local rigidity: every homeomorphism of M preserving the realized transport geometry is a C^1-diffeomorphism.

(S10) Quadratic determinacy and causal-diamond growth: along the observer-side filtration inherited from chapter 10 and interleaved with the refinement tower in chapter 11,

$$\mu(D_T) \asymp T^2,$$

and equivalently, after reparameterizing by the corresponding refinement scale $L := k(T)$,

$$\mu(D_L) \asymp L^4.$$

The question is therefore the following.

> Which macroscopic compatibility laws are admissible for the realized metric geometry determined by the closed relational stack?

The answer is that, in the metric-only regime of the closed stack, the only admissible law is the Einstein law with cosmological term.

Theorem 16.1.1 (Macroscopic compatibility classification)**.** *Under the standing principle Standing Principle 1, let*

$$E : \mathcal{G} \to \Gamma(S^2T^*M)$$

be an admissible macroscopic operator in the sense of Theorem 16.2.1. Then there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$E_{\mu\nu}(g) = \alpha\, G_{\mu\nu}(g) + \beta\, g_{\mu\nu}.$$

Consequently, every admissible macroscopic compatibility law

$$E(g) = 0$$

is equivalent, when $\alpha \neq 0$, to

$$G_{\mu\nu}(g) + \Lambda g_{\mu\nu} = 0, \qquad \Lambda := \frac{\beta}{\alpha}.$$

The proof proceeds in five steps.

(i) finite comparison-preserving partial symmetries extend globally;

(ii) nontrivial automorphisms can be localized to arbitrary clopen cylinders in the Stone inverse limit;

(iii) under realization, those localized symmetries preserve the full realized transport geometry, including, under the inherited second-jet faithfulness condition, the pointwise first-variation pairing carried by the realized degree–2 channel attached to the stabilized quadratic carrier, and hence induce local C^1-diffeomorphisms of the realized manifold preserving the metric pairings g_p;

(iv) finite comparison data at a realized point reconstructs the quadratic form on the tangent space, and the induced tangent action realizes the full orthogonal group $O(T_pM, g_p)$;

(v) in dimension 4, every admissible local second-order symmetric metric operator is a linear combination of G and g, and the divergence identity then follows from the contracted Bianchi identity.

We then strengthen the conclusion by showing that the cosmological term is not an external order–0 residue. It is the scalar projection of the realized degree–2 channel attached to the stabilized quadratic carrier, read under the inherited second-jet faithfulness condition through the same intrinsic obstruction mechanism that supports the Einstein branch.

16.2 Stack boundary and admissible macroscopic operators

Definition 16.2.1 (Admissible macroscopic operator)**.** Let $\mathcal{G}$ denote the realized metric class on M. A map

$$E : \mathcal{G} \to \Gamma(S^2T^*M)$$

is called *admissible* if the following properties hold.

(A1) **Locality of order at most 2.** For every $x \in M$, the value $E(g)(x)$ depends only on the 2-jet $j^2_x g$.

(A2) **Metric-only dependence.** The operator depends only on the realized metric g.

(A3) **Symmetry.** For every $g \in \mathcal{G}$,

$$E_{\mu\nu}(g) = E_{\nu\mu}(g).$$

(A4) **Realized local symmetry invariance.** Whenever Φ is a realized local symmetry arising from a cylinder-supported automorphism of S_∞, one has

$$E(\Phi^* g) = \Phi^* E(g).$$

(A5) **Concrete second-order tensoriality.** In local coordinates, each component $E_{\mu\nu}(g)$ is a finite sum of complete contractions built from

$$g_{\alpha\beta}, \qquad (g^{-1})^{\alpha\beta}, \qquad \partial_\gamma g_{\alpha\beta}, \qquad \partial_{\gamma\delta} g_{\alpha\beta},$$

with smooth coefficient functions.

Remark 16.2.2. Condition (A4) supplies the microscopic symmetry input determined by the stack. The classification argument below does not require density in the full diffeomorphism group. What it requires is the tangent isotropy determined by realized local comparison symmetries.

Remark 16.2.3 (Local algebraicity of admissible operators)**.** By (A1), (A2), and (A5), the value $E(g)(p)$ depends only on the 2-jet of g at p, and after passing to normal coordinates at p, that dependence is an algebraic construction from the algebraic curvature tensor and the metric. Thus once the orthogonal isotropy at p is identified, the classification reduces to ordinary orthogonal invariant theory.

16.3 Finite typing and extension from the quantifier boundary

Definition 16.3.1 (Finite comparison constraints)**.** Let $F \subset U$ be finite. A finite directed comparison pattern over F is a specification of bits

$$(\pi^{\rightarrow}_{c,f}, \pi^{\leftarrow}_{c,f}) \in \{0,1\} \times \{0,1\} \qquad (c \in \mathcal{C},\ f \in F).$$

Its realization set is

$$X(F; \pi^{\rightarrow}, \pi^{\leftarrow}) := \Big\{ u \in U : c(u,f) = \pi^{\rightarrow}_{c,f},\ c(f,u) = \pi^{\leftarrow}_{c,f} \text{ for all } (c,f) \Big\}.$$

Lemma 16.3.2 (Typing lemma)**.** *For every finite set $F \subset U$ and every finite pattern $(\pi^{\rightarrow}, \pi^{\leftarrow})$, the realization set*

$$X(F; \pi^{\rightarrow}, \pi^{\leftarrow})$$

belongs to some finite-stage Boolean algebra B_k.

Proof. Each atomic condition $c(u, f) = 1$, $c(u, f) = 0$, $c(f, u) = 1$, or $c(f, u) = 0$ defines a comparison half-space. The realization set of the finite pattern is therefore a finite Boolean combination of finitely many such half-spaces. By construction of the refinement tower, every finitely generated Boolean subalgebra of B_∞ is contained in some finite stage B_k. Hence

$$X(F; \pi^{\rightarrow}, \pi^{\leftarrow}) \in \mathsf{B}_k$$

for some k. □

Theorem 16.3.3 (Extension from the quantifier boundary)**.** *Let*

$$\sigma : D \to D'$$

be a finite partial bijection such that

$$c(x, y) = c(\sigma(x), \sigma(y)) \qquad (x, y \in D,\ c \in \mathcal{C}).$$

Then σ extends to an element

$$\phi \in \mathsf{G} = \mathrm{Aut}(U, \mathcal{C}).$$

Proof. Fix a well-order of U, and perform a back-and-forth construction. Assume inductively that a finite comparison-preserving partial bijection

$$\sigma_n : D_n \to D'_n$$

has already been constructed. *Forth step.* Choose $u \in U \setminus D_n$. We seek $v \in U \setminus D'_n$ such that for every $x \in D_n$ and every $c \in \mathcal{C}$,

$$c(u, x) = c(v, \sigma_n(x)), \qquad c(x, u) = c(\sigma_n(x), v).$$

This prescribes a finite comparison pattern over the witness set D'_n. By Theorem 16.3.2, the set of realizations of this pattern lies in some B_k. Suppose this realization set were empty. Then the finitely many excluded comparison cells determined by the required pattern would cover U, contradicting the quantifier boundary assumption (S5). Hence the realization set is nonempty. Because only finitely many points have already been used, we may choose $v \notin D'_n$. *Back step.* Apply the same argument to the inverse partial bijection in order to extend the codomain side by one point. Proceeding transfinitely along the chosen well-order yields a total bijection

$$\phi : U \to U$$

preserving every comparison in $\mathcal{C}$. Hence

$$\phi \in \mathrm{Aut}(U, \mathcal{C}) = \mathsf{G}.$$

□

16.4 Cylinder-local automorphism completeness on the Stone limit

Definition 16.4.1 (Stone cylinders)**.** For $b \in \mathsf{B}_\infty$, define

$$\widehat{b} := \{p \in S_\infty : \ b \in p\}.$$

A *clopen cylinder* is a set of this form.

Theorem 16.4.2 (Cylinder-local completion)**.** *For every nonempty clopen cylinder $E \subseteq S_\infty$, there exists a nontrivial automorphism $\phi \in \mathsf{G}$ such that the induced Stone homeomorphism $\widehat{\phi}$ satisfies*

$$\widehat{\phi}|_{S_\infty \setminus E} = \mathrm{id}\,.$$

Proof. Write $E = \widehat{b}$ for some $b \in \mathsf{B}_\infty$. Because b belongs to the direct-limit Boolean algebra, it is decided by finitely many generators from some stage B_k. Let W be a finite witness family supporting those generators. Choose an ultrafilter $p \in E$. Since E is nonempty and clopen, and since finite-stage refinement separates finite comparison protocols, there exists another ultrafilter $q \in E$, $q \neq p$, which agrees with p on every generator supported outside W, but differs on the local witness pattern inside W. Realize the relevant finite local data of p and q by a finite witness configuration $D \subset U$. Define a finite partial bijection

$$\sigma : D \to D$$

which fixes all witnesses outside W and permutes the witnesses in W so as to interchange the local p-pattern and q-pattern. By construction, σ preserves all comparisons on D. Indeed, outside W nothing changes, while inside W the permutation has been chosen precisely so that the finite comparison type is carried to an equivalent one. Hence, by Theorem 16.3.3, σ extends to an automorphism

$$\phi \in \mathsf{G}.$$

Because σ is the identity on all data outside the support of b, the induced Stone action fixes $S_\infty \setminus E$ pointwise. Because σ exchanges the local patterns of p and q, the induced action is nontrivial on E. □

16.5 Realized local symmetries and tangent isotropy

Lemma 16.5.1 (Realized transport geometry is preserved by comparison automorphisms)**.** *Let*

$$\phi \in \mathsf{G} = \mathrm{Aut}(U, \mathcal{C}),$$

and let

$$\widehat{\phi} : S_\infty \to S_\infty$$

denote the induced Stone homeomorphism. Under the realization map

$$\iota : S_\infty \to M,$$

the induced homeomorphism of M preserves the realized transport geometry, including the realized degree–2 carrier and, under the inherited second-jet faithfulness condition, the pointwise first-variation pairing carried by the realized degree–2 channel attached to the stabilized quadratic carrier. Equivalently, it preserves both the realized curvature channel and, under that same inherited second-jet faithfulness condition, the metric pairings g_p.

Proof. By transport closure, every admissible microscopic dynamics is an element of G. Hence every $\phi \in \mathsf{G}$ acts functorially on the intrinsic transport filtration and therefore preserves each stabilized graded layer. In particular, ϕ preserves the intrinsic quadratic carrier

$$\mathcal{K}_\infty \subset F^2/F^3.$$

It also preserves the intrinsic first-variation transport data, because that data is reconstructed from the same closed comparison transport geometry. By (S7), realization carries this preserved quadratic package forward only through the repaired interface statement: under the inherited second-jet faithfulness condition, nonzero stabilized square classes are equivalent to nonzero realized curvature. By (S8), the first-variation space realizes as the tangent space and, under that same inherited second-jet faithfulness condition, the preserved first-variation pairing on the realized degree–2 channel attached to the stabilized quadratic carrier is identified with the Lorentz metric pairing on T_pM. Functoriality of realization therefore implies that the induced map on the realized locus preserves the realized curvature channel and, under the inherited second-jet faithfulness condition, the realized metric pairing carried by that realized degree–2 channel. □

Theorem 16.5.2 (Local realized symmetry theorem)**.** *Let $B \subset M$ be a coordinate ball. For every sufficiently small clopen cylinder*

$$E \subset S_\infty \qquad \text{with} \qquad \iota(E) \subset B,$$

there exists a nontrivial C^1-diffeomorphism

$$\Phi_E : M \to M$$

induced by a comparison automorphism, whose deviation from the identity is confined to B, and such that

$$E(\Phi_E^* g) = \Phi_E^* E(g)$$

for every admissible macroscopic operator E.

Proof. By Theorem 16.4.2, choose

$$\phi \in \mathsf{G}$$

whose Stone action is supported in E and is nontrivial there. By Theorem 16.5.1, the induced homeomorphism of M preserves the full realized transport geometry, hence in particular the metric pairings g_p. By the rigidity input (S9), that induced homeomorphism is a C^1-diffeomorphism; call it Φ_E. Because the Stone action is trivial off E, the only nontrivial realized effect occurs inside $\iota(E) \subset B$. Finally, admissibility condition (A4) gives

$$E(\Phi_E^* g) = \Phi_E^* E(g)$$

for every admissible macroscopic operator E. □

Definition 16.5.3 (Local realized isotropy at a point)**.** Let $p \in M$. Define

$$\mathrm{Iso}_p^{\mathrm{loc}} := \Big\{ d\Phi_p \in \mathrm{GL}(T_pM) : \Phi \text{ arises from a realized local symmetry and } \Phi(p) = p \Big\}.$$

16.6 Quadratic reconstruction from finite comparison data

Definition 16.6.1 (Stabilized degree–2 comparison data at a point)**.** Fix $p \in M$, and identify the first-variation arena V_p with T_pM via the smooth realization theorem (Theorem 17.9.1(ii) together with Section 17.12). For a sufficiently small realized neighborhood of p, the *stabilized degree–2 comparison data near p* consists of the finite comparison relations detected by the stabilized quadratic carrier

$$\mathcal{K}_\infty \subset F^2/F^3$$

on local comparison directions converging to tangent directions at p.

Remark 16.6.2. By Proposition 17.2.2, the stack canonically determines a symmetric bilinear form

$$B_p : V_p \times V_p \to \mathbb{R}.$$

By Theorem 17.9.1(ii), the first-variation arena realizes as the tangent space, and, under the inherited second-jet interface hypothesis, the relevant pairing on the realized degree–2 channel attached to the stabilized quadratic carrier is identified there with the Lorentz metric pairing

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

Thus the pointwise quadratic form is not auxiliary structure: it is the realized form of the intrinsic first-variation transport pairing.

Theorem 16.6.3 (Quadratic reconstruction from finite comparison data)**.** *Let $p \in M$, and set*

$$V := T_pM.$$

There exists a canonical map

$$\mathfrak{q}_p : \Big\{ \textit{stabilized finite degree–2 comparison data near } p \Big\} \longrightarrow \mathrm{Sym}^2(V^*)$$

characterized by the property that under smooth realization it sends the intrinsic first-variation comparison data, through that realized degree–2 channel attached to the stabilized quadratic carrier, to the Lorentz metric pairing

$$g_p.$$

Moreover, this map is injective: if two symmetric bilinear forms

$$q, \widetilde{q} \in \mathrm{Sym}^2(V^*)$$

induce identical stabilized finite degree–2 comparison data in every sufficiently small neighborhood of p, then

$$q = \widetilde{q}.$$

In particular, the realized metric pairing g_p is uniquely reconstructed from finite comparison data.

Proof. We divide the proof into four steps. **Step 1: intrinsic degree–2 data determines first-variation pairings.** By Proposition 17.2.2, for each point of the first-variation locus the closed stack canonically determines a symmetric bilinear pairing

$$B_p : V_p \times V_p \to \mathbb{R}.$$

This pairing is part of the intrinsic transport geometry. It is read off from the degree–2 transport layer: the stabilized carrier

$$\mathcal{K}_\infty \subset F^2/F^3$$

encodes precisely the quadratic comparison relations between local comparison directions. Therefore, once local comparison directions have been chosen, the stabilized degree–2 comparison data determines the values of the first-variation pairing on those directions. **Step 2: passage to the realized tangent space.** By Theorem 17.9.1(ii), together with the identification of Section 17.12, the first-variation arena V_p realizes as the tangent space T_pM, and the intrinsic pairing B_p realizes as the Lorentz metric pairing g_p. Thus for realized tangent directions $v, w \in V = T_pM$, any local comparison directions converging to v and w determine, through the stabilized degree–2 carrier, a well-defined quadratic pairing whose realized value is exactly

$$g_p(v, w).$$

This defines the map $\mathfrak{q}_p$: given stabilized finite degree–2 comparison data, assign to each pair (v, w) the corresponding realized first-variation value. Because the underlying pairing is symmetric and bilinear, the resulting assignment lies in $\mathrm{Sym}^2(V^*)$. **Step 3: well-definedness.** We must check that the assigned value does not depend on the particular finite approximation. Choose $v, w \in V$, and let

$$(v_\nu, w_\nu) \qquad \text{and} \qquad (v'_\mu, w'_\mu)$$

be two sufficiently fine families of local comparison directions converging to v and w. Since the carrier $\mathcal{K}_\infty$ is stabilized, the degree–2 comparison relations are eventually constant at sufficiently fine stage. Hence both approximating families determine the same first-variation value. After realization, both therefore give the same number

$$g_p(v, w).$$

So $\mathfrak{q}_p$ is well-defined. **Step 4: injectivity.** Let

$$q, \widetilde{q} \in \mathrm{Sym}^2(V^*)$$

induce identical stabilized finite degree–2 comparison data in every sufficiently small neighborhood of p. We must prove

$$q = \widetilde{q}.$$

Fix arbitrary $v, w \in V$. Choose local realized comparison directions converging to v and w. Because the stabilized finite degree–2 comparison data induced by q and $\widetilde{q}$ agree at every sufficiently fine stage, the corresponding first-variation values assigned to the pair (v, w) also agree. Therefore

$$q(v, w) = \widetilde{q}(v, w).$$

Since $v, w \in V$ were arbitrary, the two bilinear forms agree on every pair of vectors. Hence

$$q = \widetilde{q}.$$

Applying this conclusion to the realized first-variation transport data shows that there is exactly one symmetric bilinear form on T_pM compatible with the stabilized finite degree–2 comparison data near p, namely the realized metric pairing g_p. This proves both the existence of $\mathfrak{q}_p$ and its injectivity. □

Theorem 16.6.4 (Canonical frame realizability)**.** *Let $p \in M$, and let*

$$(e_1, \ldots, e_n)$$

be an orthonormal frame of T_pM. Then there exists a finite comparison configuration localized near p, together with a distinguished ordering of its realized tangent directions, whose stabilized degree–2 comparison data reconstructs exactly the Gram matrix

$$g_p(e_i, e_j) = \delta_{ij}.$$

Moreover, if two such ordered configurations realize two orthonormal frames with the same Gram matrix, then there exists a comparison-preserving finite bijection carrying the first ordered configuration to the second.

Proof. Fix an orthonormal frame $(e_1, \ldots, e_n)$. Choose local realized comparison directions converging to these tangent directions. At sufficiently fine finite stage, the induced stabilized degree–2 comparison data determine, by Theorem 16.6.3, the unique quadratic form g_p on the span of these directions. Because the frame is orthonormal, the reconstructed Gram matrix is δ_{ij}. This yields the desired ordered finite configuration. Now let two such ordered configurations realize orthonormal frames with the same Gram matrix. Since both ordered configurations induce the same stabilized finite degree–2 comparison data, they have the same finite comparison type. Therefore the order-preserving identification of one configuration with the other is comparison-preserving. This gives the required finite bijection. □

Theorem 16.6.5 (Isotropy completeness)**.** *Let $p \in M$, and set*

$$V := T_pM.$$

Then

$$\mathrm{Iso}_p^{\mathrm{loc}} = O(V, g_p).$$

Proof. We prove the two inclusions separately. **Step 1:** $\mathrm{Iso}_p^{\mathrm{loc}} \subseteq O(V, g_p)$**.** Let

$$A \in \mathrm{Iso}_p^{\mathrm{loc}}.$$

By definition, there exists a realized local symmetry Φ fixing p such that

$$A = d\Phi_p.$$

By Theorem 16.5.1, every realized local symmetry preserves the full realized transport geometry. In particular, it preserves the pointwise first-variation pairing, hence the metric pairing g_p. Therefore

$$g_p(Au, Av) = g_p(u, v) \qquad (u, v \in V),$$

so $A \in O(V, g_p)$. Thus

$$\mathrm{Iso}_p^{\mathrm{loc}} \subseteq O(V, g_p).$$

Step 2: $O(V, g_p) \subseteq \mathrm{Iso}_p^{\mathrm{loc}}$**.** Let

$$A \in O(V, g_p)$$

be arbitrary. Choose an orthonormal frame

$$(e_1, \ldots, e_n)$$

of V, and define

$$f_i := Ae_i.$$

Because A is orthogonal, the frame

$$(f_1, \ldots, f_n)$$

is also orthonormal. By Theorem 16.6.4, there exist ordered finite comparison configurations localized in arbitrarily small neighborhoods of p realizing the frames (e_i) and (f_i).

Because A is orthogonal, both frames have the same Gram matrix δ_{ij}. Hence, again by Theorem 16.6.4, there exists a comparison-preserving finite bijection carrying the first ordered configuration to the second. By Theorem 16.3.3, this finite bijection extends to a global comparison automorphism

$$\phi \in \mathsf{G}.$$

By Theorem 16.5.2, the corresponding realized map is a local C^1-diffeomorphism Φ fixing p. Its differential carries the realized tangent directions of the first frame to those of the second. Hence

$$d\Phi_p(e_i) = f_i = Ae_i \qquad (i = 1, \dots, n).$$

Since the vectors e_i form a basis of V, it follows that

$$d\Phi_p = A.$$

Therefore $A \in \mathrm{Iso}_p^{\mathrm{loc}}$, so

$$O(V, g_p) \subseteq \mathrm{Iso}_p^{\mathrm{loc}}.$$

Combining the two inclusions yields

$$\mathrm{Iso}_p^{\mathrm{loc}} = O(V, g_p).$$

□

Corollary 16.6.6 (Full orthogonal equivariance)**.** *Every admissible local second-order metric operator is $O(T_pM, g_p)$-equivariant at p.*

Proof. Let

$$F_p : J_p^2(\mathcal{G}) \to S^2T_p^*M$$

be the pointwise map corresponding to an admissible operator E. By (A4), F_p is equivariant under every realized local symmetry fixing p, hence under $\mathrm{Iso}_p^{\mathrm{loc}}$. By Theorem 16.6.5,

$$\mathrm{Iso}_p^{\mathrm{loc}} = O(T_pM, g_p),$$

so F_p is $O(T_pM, g_p)$-equivariant. □

16.7 Dimension from intrinsic causal-diamond growth

Theorem 16.7.1 (Dimension from causal diamonds)**.** *Assume that the intrinsic causal-diamond growth law satisfies*

$$\mu(D_L) \asymp L^4,$$

and that the realized small causal-diamond volume is asymptotically comparable to that intrinsic count. Then

$$\dim M = 4.$$

Proof. Let $n := \dim M$. In an n-dimensional smooth Lorentzian manifold, the volume of a sufficiently small causal diamond of linear scale L has asymptotic expansion

$$\mathrm{Vol}(D_L) = C_n L^n + o(L^n), \qquad C_n > 0,$$

where C_n is the model tangent-cone constant. By hypothesis, this realized small-diamond volume is asymptotically comparable to the stabilized intrinsic count, and the latter satisfies

$$\mu(D_L) \asymp L^4.$$

Hence the growth exponents must agree. Therefore $n = 4$. □

16.8 Second-order local tensor classification

Theorem 16.8.1 (Second-order classification in dimension 4)**.** *Assume* $\dim M = 4$. *Let*

$$E : \mathcal{G} \to \Gamma(S^2T^*M)$$

be admissible. Then there exist constants $a, b, c \in \mathbb{R}$ *such that*

$$E_{\mu\nu} = a\ \mathrm{Ric}_{\mu\nu} + b\, g_{\mu\nu}\,\mathrm{Scal} + c\, g_{\mu\nu}.$$

If one writes

$$a = \alpha, \qquad b = -\frac{\alpha}{2}, \qquad c = \beta,$$

then equivalently

$$E_{\mu\nu} = \alpha\, G_{\mu\nu} + \beta\, g_{\mu\nu}.$$

Proof. By (A1)–(A5), the value $E(g)(p)$ is a natural second-order tensorial expression depending only on the 2-jet of g. By Theorem 16.6.6, this dependence is $O(T_pM, g_p)$-equivariant. Therefore $E(g)(p)$ is an $O(V, g_p)$-equivariant algebraic function of the algebraic curvature tensor and the metric, where

$$V := T_pM.$$

Pass to normal coordinates at p. Then the first derivatives of g vanish, so the only second-order data are the algebraic curvature tensor and its contractions. By classical orthogonal invariant theory for natural second-order symmetric tensors, the only such tensors are linear combinations of

$$\mathrm{Ric}_{\mu\nu}, \qquad g_{\mu\nu}\,\mathrm{Scal}, \qquad g_{\mu\nu}.$$

Hence there exist constants $a, b, c \in \mathbb{R}$ such that

$$E_{\mu\nu} = a\ \mathrm{Ric}_{\mu\nu} + b\, g_{\mu\nu}\,\mathrm{Scal} + c\, g_{\mu\nu}.$$

Writing

$$\alpha := a, \qquad \beta := c, \qquad b := -\frac{\alpha}{2},$$

one obtains

$$a \operatorname{Ric}_{\mu\nu} + b\, g_{\mu\nu} \operatorname{Scal} = \alpha \left(\operatorname{Ric}_{\mu\nu} - \frac{1}{2} \operatorname{Scal}\ g_{\mu\nu} \right) = \alpha\, G_{\mu\nu}.$$

Therefore

$$E_{\mu\nu} = \alpha\, G_{\mu\nu} + \beta\, g_{\mu\nu}.$$

□

Corollary 16.8.2 (Divergence identity as a consequence)**.** *If E is admissible, then*

$$\nabla_\mu E^{\mu\nu}(g) = 0$$

for every realized metric g.

Proof. By Theorem 16.8.1,

$$E_{\mu\nu} = \alpha\, G_{\mu\nu} + \beta\, g_{\mu\nu}.$$

Since

$$\nabla^\mu G_{\mu\nu} = 0$$

by the contracted Bianchi identity and

$$\nabla^\mu g_{\mu\nu} = 0$$

by metric compatibility of the Levi–Civita connection, one obtains

$$\nabla^\mu E_{\mu\nu} = 0.$$

□

16.9 The scalar channel of the stabilized quadratic carrier

Section 16.8 identifies $\beta g_{\mu\nu}$ as the unique order–0 symmetric natural residue. We now show that this coefficient is not external to the transport picture. It is the scalar projection on the realized degree–2 channel attached to the stabilized quadratic carrier $\mathcal{K}_\infty \subset F^2/F^3$ under the inherited second-jet faithfulness condition.

Definition 16.9.1 (Realized algebraic curvature module)**.** Let $p \in M$, and set $V := T_pM$. Write $\mathcal{R}(V) \subset V^{*\otimes 4}$ for the space of algebraic curvature tensors. Under the orthogonal group $O(V, g_p)$, there is the standard equivariant decomposition

$$\mathcal{R}(V) = \mathcal{W}(V) \oplus \mathcal{E}(V) \oplus \mathcal{S}(V),$$

where $\mathcal{W}(V)$ is the Weyl summand, $\mathcal{E}(V)$ is the trace-free Ricci summand, and $\mathcal{S}(V)$ is the scalar summand.

Remark 16.9.2. The scalar summand $\mathcal{S}(V)$ is one-dimensional, generated by $g_p \wedge g_p$.

Definition 16.9.3 (Canonical scalar channel)**.** Assume the inherited second-jet faithfulness condition in a compatible smooth realization, and write

$$\mathsf{Jet}_2(\mathcal{K}_\infty) \subseteq \mathcal{R}(V)$$

for the realized degree–2 channel attached to the stabilized quadratic carrier. Let

$$\mathrm{pr}_{\mathrm{scal}} : \mathcal{R}(V) \to \mathcal{S}(V)$$

be the $O(V, g_p)$-equivariant projection onto the scalar summand. Fix a nonzero invariant linear functional

$$\Phi : \mathcal{S}(V) \to \mathbb{R}.$$

Define the canonical scalar channel on that realized degree–2 channel by

$$\beta_\triangle := \Phi \circ \mathrm{pr}_{\mathrm{scal}}\big|_{\mathsf{Jet}_2(\mathcal{K}_\infty)}.$$

Lemma 16.9.4 (Uniqueness of the scalar channel)**.** *The scalar channel $\beta_\triangle$ is well-defined up to an overall nonzero normalization constant.*

Proof. The space $\mathcal{S}(V)$ is one-dimensional. Hence any two nonzero invariant linear functionals on $\mathcal{S}(V)$ differ by multiplication by a nonzero scalar. Therefore the composite

$$\Phi \circ \mathrm{pr}_{\mathrm{scal}}\big|_{\mathsf{Jet}_2(\mathcal{K}_\infty)}$$

is unique up to such a normalization. □

Theorem 16.9.5 (Cosmological coefficient from the quadratic carrier)**.** *There exists a constant $\kappa \neq 0$ such that*

$$\beta = \kappa\, \beta_\triangle.$$

Proof. By Theorem 16.8.1, the only order–0 symmetric natural term in an admissible macroscopic operator is

$$\beta g_{\mu\nu}.$$

Thus β is the unique surviving macroscopic scalar coefficient. On the other hand, under the inherited second-jet interface hypothesis, nonzero stabilized square classes in the stabilized quadratic carrier $\mathcal{K}_\infty \subset F^2/F^3$ are equivalent to nonzero realized curvature, and $\beta_\triangle$ is precisely the canonical scalar projection on the realized degree–2 channel attached to that stabilized quadratic carrier. Both β and $\beta_\triangle$ are scalar-valued, local, and attached to the same one-dimensional invariant scalar channel. Hence they differ by a nonzero proportionality constant:

$$\beta = \kappa\, \beta_\triangle.$$

□

Corollary 16.9.6 (Vanishing criterion)**.**

$$\beta = 0 \iff \beta_\triangle = 0.$$

In particular, the cosmological term vanishes exactly when the scalar projection of the stabilized quadratic carrier vanishes.

Proof. Immediate from Theorem 16.9.5, since $\kappa \neq 0$. ☐

Remark 16.9.7. Thus the two surviving macroscopic terms come from the same intrinsic obstruction package carried by the stabilized quadratic transport layer:

$$\alpha G_{\mu\nu} \quad \text{is the trace-free channel on the realized degree–2 channel,}$$

while

$$\beta g_{\mu\nu} \quad \text{is the scalar channel on that same realized degree–2 channel.}$$

The full admissible macroscopic compatibility law is therefore generated by that intrinsic obstruction package, read under the inherited second-jet faithfulness condition, and by no additional independent source.

16.10 Einstein compatibility determination

Proof of Theorem 16.1.1. By Theorem 16.3.3, finite comparison-preserving partial symmetries extend globally. By Theorem 16.4.2, the comparison symmetry group G is cylinder-locally complete on S_∞. By Theorem 16.5.1 and Theorem 16.5.2, these localized Stone symmetries induce local realized C^1-diffeomorphisms. By Theorem 16.6.3, Theorem 16.6.4, and Theorem 16.6.5, their tangent action realizes the full orthogonal isotropy needed for pointwise invariant classification. By Theorem 16.7.1, one has

$$\dim M = 4.$$

Hence Theorem 16.8.1 applies and yields

$$E_{\mu\nu}(g) = \alpha\, G_{\mu\nu}(g) + \beta\, g_{\mu\nu}.$$

Finally, by Theorem 16.9.5, the cosmological coefficient β is the scalar projection of the same stabilized quadratic carrier

$$\mathcal{K}_\infty \subset F^2/F^3$$

whose realized degree–2 channel, under the inherited second-jet faithfulness condition, carries the same repaired curvature bridge: nonzero stabilized square classes are equivalent to nonzero realized curvature. Thus both surviving macroscopic terms arise from the same intrinsic obstruction mechanism. If the admissible compatibility law is

$$E(g) = 0,$$

then

$$\alpha\, G_{\mu\nu}(g) + \beta\, g_{\mu\nu} = 0.$$

When $\alpha \neq 0$, divide by α to obtain

$$G_{\mu\nu}(g) + \Lambda g_{\mu\nu} = 0, \qquad \Lambda := \frac{\beta}{\alpha}.$$

□

16.11 Structural summary

The structural chain proved in this chapter is

$$\begin{aligned} \text{quantifier boundary} &\Longrightarrow \text{extension of finite partial symmetries} \\ &\Longrightarrow \text{cylinder-local Stone symmetries} \\ &\Longrightarrow \text{realized local } C^1\text{-symmetries} \\ &\Longrightarrow \text{quadratic reconstruction from finite comparison data} \\ &\Longrightarrow \text{canonical frame realizability} \\ &\Longrightarrow \text{full orthogonal isotropy} \\ &\Longrightarrow O(T_pM, g_p) \\ &\Longrightarrow \dim M = 4 \\ &\Longrightarrow \text{second-order local classification} \\ &\Longrightarrow E_{\mu\nu} = \alpha G_{\mu\nu} + \beta g_{\mu\nu}. \end{aligned}$$

with

$$\begin{aligned} \alpha G_{\mu\nu} &= \text{trace-free channel on the realized degree–2 channel} \\ &\quad\ \text{attached to the stabilized quadratic carrier, read under the} \\ &\quad\ \text{inherited second-jet faithfulness condition,} \\ \beta g_{\mu\nu} &= \text{scalar channel on that same realized degree–2 channel.} \end{aligned}$$

Accordingly, the closed relational stack determines the Einstein compatibility law and identifies both surviving macroscopic terms with one intrinsic obstruction package, read on the realized degree–2 channel attached to

$$F^2/F^3.$$

under the inherited second-jet faithfulness condition.

16.12 Transition to connection-first reconstruction

This chapter fixes the admissible macroscopic compatibility law in the metric-only regime of the closed stack. Within that regime one obtains

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

This is not a terminal halt of the program. It is the boundary datum for the next reconstruction layer. To pass from vacuum compatibility to sourced dynamics, one must identify which additional intrinsic sectors can couple without violating closure, quotient descent, and transport visibility. That continuation starts in chapter 17, where the same quadratic carrier is recast in connection-first form and prepared for the downstream phase and matter sectors.

16.13 Conclusion

Within the metric-only regime of the closed stack, the admissible macroscopic compatibility law is uniquely fixed, and its surviving tensor channels are both traced to the same stabilized quadratic obstruction source.

Accordingly, chapter 17 starts from this Einstein boundary law, reconstructs the same content in connection-first variables, and opens the intrinsic route to the phase dynamics of chapter 18 and the sourced matter couplings of chapter 19.

Chapter 17

Connection-First Reconstruction of Geometry

17.1 Introduction

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the connection-first realization endpoint of the transport stack determined by item (SP5). Recasting the transport-to-curvature results, it develops a connection-first reconstruction that bridges the intrinsic stack to chapter 18. Chapters 2, 6 and 13 to 16 determine the intrinsic geometric content of the closed stack up to first-order gauge. Closure determines quotient semantics. Admissible state content is exactly quotient content on

$$\mathsf{Phys} := X/G.$$

Thus no invariant state datum remains except what descends through the orbit projection

$$\pi : X \to \mathsf{Phys}.$$

The classification of enrichment then shows that every non-quotient enrichment is exhausted by two loci, possibly with both and with no third:

(a) object-level representative choice;

(b) morphism-level transport.

Object-level enrichment is noncanonical. Once quotient semantics has been determined, the only possible intrinsic carrier of geometry is therefore transport. Spacetime interleaving next constructs the canonical two-parameter object

$$ST := \varprojlim_{k,t} \big(S_k/F_t^{(k)}\big),$$

which is the first object in the stack on which spatial refinement and temporal coarsening coexist in a single intrinsic inverse-limit construction. Finally, the obstruction theory proves that the first visible intrinsic transport residue is the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3,$$

and that, under the inherited second-jet faithfulness condition, nonzero stabilized square classes are equivalent to nonzero realized curvature in any compatible smooth realization. Thus geometry is no longer conjectural. It is already determined. The remaining question is to determine the form in which geometry is intrinsic before auxiliary choices are made. The answer is connection-first geometry. Indeed, an atlas is first-order coordinate data. But first-order coordinate data cannot be intrinsic in a closed system: a coordinate assignment requires representative choice in orbit fibres, hence lies in the object locus, and the object locus has already been proved noncanonical. What survives canonically after quotient descent is not coordinates but transport. Accordingly, the intrinsic geometric object determined by the stack is not a coordinate manifold but a connection-type structure: a spacetime precursor, a family of first-variation arenas, a first-order transport law defined only up to gauge, and a first tensorial obstruction carried by the quadratic carrier

$$\mathcal{K}.$$

The chapter proves, in order:

(i) that no coordinate atlas can be canonically reconstructed from the closed comparison structure;

(ii) that the regular locus of ST carries a canonical family of four-dimensional first-variation spaces and a first-order transport law;

(iii) that these data determine an intrinsic connection class on the first-variation bundle;

(iv) that the first tensorial residue of this connection class is precisely the quadratic carrier

$$\mathcal{K} \simeq F^2/F^3;$$

(v) that the entire transport filtration

$$F^1 \supseteq F^2 \supseteq F^3 \supseteq \cdots$$

is the intrinsic jet hierarchy of this connection-first geometry;

(vi) that whenever a compatible smooth realization exists, the intrinsic connection-first geometry realizes as ordinary Lorentzian connection geometry and, under the inherited second-jet faithfulness condition, nonzero stabilized square classes in $\mathcal{K}$ are equivalent to nonzero realized curvature in that realization.

Thus the conclusion is exact:

coordinates belong to gauge, transport does not.

Geometry in a closed comparison system is intrinsically connection-first.

17.2 The spacetime precursor and the first-variation family

Recall from Chapter 14 that the refinement tower and the observer-induced temporal quotient filtration determine the spacetime object

$$ST := \varprojlim_{k,t} \big(S_k / F_t^{(k)}\big).$$

By construction, ST is the first object in the stack in which the two structural directions

$$k \uparrow \text{ (spatial refinement)}, \qquad t \uparrow \text{ (temporal coarsening)}$$

coexist inside one intrinsic inverse-limit object. At this stage ST is canonical. What remains observer-relative is the observer reduction and the finite levels to which it descends. No coordinate structure has been introduced. The causality chapter already isolates the locus on which first-order geometric data are nondegenerate.

Definition 17.2.1 (Regular locus)**.** Define

$$ST^{\mathrm{reg}} \subseteq ST$$

to be the subset of points $p \in ST$ for which the first-variation arena V_p exists and satisfies

$$\dim V_p = 4.$$

Proposition 17.2.2 (Pointwise first-variation geometry)**.** *For every $p \in ST^{\mathrm{reg}}$, the stack canonically determines:*

(i) *a real vector space V_p of dimension 4;*

(ii) *a symmetric bilinear form*

$$B_p : V_p \times V_p \to \mathbb{R};$$

(iii) *under the causal-cone hypotheses of Chapter 14, a Lorentzian signature condition*

$$\mathrm{sign}(B_p) \in \{(1,3),(3,1)\}.$$

Proof. This is exactly the pointwise conclusion proved in Section 14.6 of Chapter 14. The definition of ST^{reg} merely isolates the locus on which that conclusion holds. □

Definition 17.2.3 (First-variation bundle)**.** Define

$$T^{(1)} ST^{\mathrm{reg}} := \bigsqcup_{p \in ST^{\mathrm{reg}}} V_p$$

with projection

$$\pi_1 : T^{(1)} ST^{\mathrm{reg}} \to ST^{\mathrm{reg}}, \qquad \pi_1(v) = p \quad (v \in V_p).$$

17.3 No intrinsic coordinates

The coordinate-manifold picture assumes canonical local coordinate assignments determined purely by the intrinsic structure of the system. We show that such assignments are structurally impossible in a closed comparison system.

Definition 17.3.1 (Intrinsic coordinate system)**.** An *intrinsic coordinate system* on a space Y constructed from the closed comparison structure is a rule assigning to every point $p \in Y$ a neighborhood $U_p \subseteq Y$ and a map

$$\phi_p : U_p \to \mathbb{R}^n$$

such that the assignment $p \mapsto (U_p, \phi_p)$ is determined functorially from the closed comparison data and is invariant under all automorphisms of the comparison structure.

Theorem 17.3.2 (Universal obstruction to intrinsic coordinates)**.** *Let*

$$(X, G, \pi : X \to \mathsf{Phys} = X/G)$$

be the quotient system determined by closure. Then no intrinsic coordinate system exists on Phys *or on any geometric object constructed solely from quotient-level state content. Equivalently, every coordinate construction necessarily depends on noncanonical gauge data.*

Proof. Closure identifies admissible state content with quotient content descending through

$$\pi : X \to \mathsf{Phys}.$$

Hence any intrinsic coordinate system on a quotient-level geometric object must be determined functorially from the quotient datum (X, G, π). Let $\mathrm{Aut}(X, G, \pi)$ denote the automorphism group of this quotient datum. Every element

$$\alpha \in \mathrm{Aut}(X, G, \pi)$$

induces an automorphism of Phys. If an intrinsic coordinate system exists, then functoriality requires the coordinate assignment to be equivariant under every such α. Thus for each $p \in \mathsf{Phys}$ and each $\alpha \in \mathrm{Aut}(X, G, \pi)$ the local coordinate maps satisfy

$$\phi_{\alpha(p)} \circ \alpha = \phi_p$$

on the common domain of definition. Now fix $p \in \mathsf{Phys}$. Because $\pi : X \to \mathsf{Phys}$ is an orbit projection, the fibre $\pi^{-1}(p)$ is a G-orbit. The group G acts transitively on that fibre, and this transitively permutes all representative choices above p. The induced symmetries extend to local automorphisms of the quotient datum. Therefore equivariance under $\mathrm{Aut}(X, G, \pi)$ determines any canonically assigned first-order local coordinate data to be invariant under this transitive representative action. But invariance under transitive representative action eliminates all distinguished first-order labels attached to representatives. In particular, no canonically chosen coordinate germ can separate local states by a representative-dependent first-order assignment. An intrinsic coordinate system would therefore fail to distinguish nearby points by locally injective coordinate maps. This contradicts the defining property of a coordinate system. Hence no intrinsic coordinate system exists. □

Corollary 17.3.3 (Coordinates are gauge)**.** *Any coordinate atlas on a geometry derived from the closed comparison system necessarily depends on auxiliary gauge choices.*

Proof. By Theorem 17.3.2, coordinates cannot be constructed functorially from the intrinsic data. Any coordinate assignment therefore requires extra choices breaking the automorphism symmetry of the quotient system. □

17.4 Transport as first-order geometry

The morphism locus already carries transport before any obstruction appears. Chapters 3 and 6 show that a representative lift is exactly representative choice together with a flat G-connection on the protocol category. The later bridge, interface, and causality chapters show that the first stable failure of this flat compositional regime appears in degree 2. This is the formal pattern of a connection with curvature. The present section extracts that pattern intrinsically.

Definition 17.4.1 (First-order transport law)**.** A *first-order transport law* on ST^{reg} is an assignment, for nearby regular points $p, q \in ST^{\mathrm{reg}}$, of linear maps

$$\tau_{q\to p} : V_q \to V_p$$

obtained from the morphism-locus transport by passage to first variation.

Proposition 17.4.2 (Transport is the unique intrinsic first-order residue)**.** *On ST^{reg}, every intrinsic geometric datum not already exhausted by quotient semantics must arise from morphism transport and not from object-level structure.*

Proof. Closure determines quotient semantics, so quotient-level state content is already complete. The two-locus classification shows that every non-quotient enrichment is exhausted by representative choice and morphism-level transport, possibly with both and with no third locus. By Theorem 17.3.2, representative choice cannot yield intrinsic first-order geometry. Therefore any intrinsic first-order residue must come from the morphism-transport locus. □

Proposition 17.4.3 (Functoriality to first order)**.** *The first-order transport law satisfies:*

(i)

$$\tau_{p\to p} = \mathrm{id}_{V_p};$$

(ii) *whenever the relevant comparisons are defined,*

$$\tau_{r\to p} = \tau_{q\to p} \circ \tau_{r\to q}$$

to first order;

(iii) *the first visible failure of strict transport composition is invisible in degree* 1 *and appears first in the quadratic carrier*

$$\mathcal{K} \simeq F^2/F^3.$$

Proof. Items (i) and (ii) are the first-order shadow of the flat transport formalism of Chapters 3 and 6: before obstruction appears, representative transport is flat and composes functorially. Item (iii) is exactly the content of the later obstruction theory. The first stable intrinsic failure of flat transport composition does not survive in degree 1; it appears first in the degree–2 quotient

$$F^2/F^3,$$

as proved in the bridge, interface, and causality chapters. □

Corollary 17.4.4 (Connection pattern)**.** *The intrinsic first-order geometry of* ST^{reg} *has the formal shape of a connection: first-order transport composes functorially, and its first tensorial obstruction occurs in degree* 2*.*

Proof. This is the combined content of Theorems 17.4.2 and 17.4.3. □

17.5 Classification of intrinsic first-order geometry

Theorem 17.4.4 identifies morphism transport as the only possible source of intrinsic first-order geometry. We now sharpen this to a classification theorem.

Definition 17.5.1 (First-order geometric residue)**.** A *first-order geometric residue* on ST^{reg} is any assignment of linear comparison data between nearby first-variation spaces

$$V_p$$

which is functorial under refinement and compatible with quotient semantics.

Theorem 17.5.2 (Classification of first-order geometric residues)**.** *Every intrinsic first-order geometric residue on* ST^{reg} *arises uniquely from morphism-locus transport. Equivalently, the space of intrinsic first-order geometric structures on* ST^{reg} *is canonically identified with the space of transport laws induced by representative lifts.*

Proof. Let R be a first-order geometric residue. By definition, R assigns linear comparison maps between nearby first-variation spaces

$$V_p.$$

Closure implies that quotient semantics already exhausts invariant state content. Therefore any additional structure is exhausted by representative choice and morphism-level transport, possibly with both and with no third enrichment locus. Object-level enrichment depends on representative choice and is therefore noncanonical. Since R is intrinsic by assumption, it cannot arise from that locus. Hence R must arise from morphism transport. Conversely, the representative-lift formalism of Chapter 6 constructs exactly such transport laws. Passing to first variation produces linear comparison maps

$$\tau_{q\to p} : V_q \to V_p$$

between nearby fibres, and these satisfy the required functoriality and quotient-compatibility conditions. Thus

$$R \longleftrightarrow \tau$$

is a bijective correspondence. Hence every intrinsic first-order geometric residue arises uniquely from morphism transport. □

Corollary 17.5.3 (Uniqueness of intrinsic first-order geometry)**.** *Up to gauge, the only intrinsic first-order geometric structure on* ST^{reg} *is the transport law induced by representative lifts.*

Proof. Immediate from Theorem 17.5.2. □

17.6 Local gauges and the intrinsic connection class

Since coordinates are gauge, first-order transport must be handled by local gauge-fixing rather than by a canonical atlas.

Definition 17.6.1 (Local flattening gauge)**.** A *local flattening gauge* at a point $p \in ST^{\mathrm{reg}}$ is a local choice of representative linearization of the first-variation family near p in which the first-order transport law is written explicitly as an affine connection-type law on V_p.

Proposition 17.6.2 (Existence and uniqueness up to gauge)**.** *Every regular point* $p \in ST^{\mathrm{reg}}$ *admits local flattening gauges, and any two such gauges differ by first-order gauge transformation.*

Proof. Existence is the local form of the representative-lift formalism. The morphism locus supplies local transport data; choosing local representatives identifies these data with first-order comparison maps between the first-variation spaces. This is exactly a local flattening gauge. If two such gauges are chosen, they differ by local change of representatives. Chapter 6 proves that changing representatives acts on transport by the gauge modification law. Passing to first variation, the two first-order transport descriptions therefore differ by first-order gauge transformation. □

Definition 17.6.3 (Intrinsic connection class)**.** The *intrinsic connection class* on ST^{reg} is the gauge equivalence class of first-order transport laws on

$$T^{(1)}ST^{\mathrm{reg}}$$

induced by all local flattening gauges.

Theorem 17.6.4 (Connection-first reconstruction)**.** *Under the standing principle Standing Principle 1, the closed-system stack canonically determines on ST^{reg}:*

(i) *the first-variation bundle*

$$T^{(1)}ST^{\mathrm{reg}} = \bigsqcup_{p \in ST^{\mathrm{reg}}} V_p;$$

(ii) *the pointwise causal type and, under the causal-cone hypotheses, the pointwise Lorentzian first-variation pairings B_p;*

(iii) *an intrinsic connection class on $T^{(1)}ST^{\mathrm{reg}}$, defined modulo first-order gauge by comparison transport.*

Proof. Item (i) is Theorem 17.2.3. Item (ii) is Theorem 17.2.2. For item (iii), Theorem 17.5.2 shows that every intrinsic first-order geometric structure on ST^{reg} arises uniquely from morphism transport. Passing to first variation produces linear comparison maps

$$\tau_{q \to p} : V_q \to V_p$$

between nearby fibres. By Theorem 17.4.3, these maps compose functorially to first order. Local representative choices produce gauges in which the transport law is written explicitly as an affine connection. By Theorem 17.6.2, different gauges differ by first-order gauge transformation. Therefore the gauge-equivalence class of such transport laws is canonically determined. This is exactly the intrinsic connection class. □

17.7 Quadratic obstruction as curvature carrier

We now identify the first tensorial residue of the connection-first geometry.

Theorem 17.7.1 (Quadratic obstruction is the intrinsic curvature carrier)**.** *For the intrinsic connection class reconstructed in Theorem 17.6.4, the first tensorial transport obstruction is the stabilized quadratic carrier*

$$\mathcal{K} \simeq F^2/F^3.$$

Equivalently, $\mathcal{K}$ is the intrinsic curvature carrier of the connection-first geometry.

Proof. By the bridge, interface, and causality chapters, the first stable failure of flat transport composition appears in the degree–2 quotient

$$F^2/F^3.$$

No tensorial degree–1 obstruction survives. The intrinsic connection class reconstructed in Theorem 17.6.4 is, by definition, the first-order geometry carried by morphism-locus transport on the first-variation bundle. Therefore its first tensorial obstruction is exactly the first stable tensorial failure of transport composition. But that failure is precisely the stabilized quadratic carrier $\mathcal{K}$. Hence $\mathcal{K}$ is the intrinsic curvature carrier of the connection-first geometry. □

Corollary 17.7.2 (Quadratic determination of curvature). *The intrinsic geometric hierarchy on ST^{reg} begins as*

$$\textit{first-order transport class} \implies \mathcal{K} \simeq F^2/F^3,$$

and, under the inherited second-jet faithfulness condition in any compatible smooth realization, nonzero stabilized square classes in $\mathcal{K}$ are equivalent to nonzero realized curvature.

Proof. The first step is Theorem 17.7.1. For the realized branch, Theorem 13.12.3 gives the inherited second-jet interface criterion: in any compatible smooth realization, nonzero stabilized square classes in $\mathcal{K}$ are equivalent to nonzero realized curvature. □

17.8 The transport filtration as an intrinsic jet tower

Section 17.7 and Theorem 17.7.1 identify the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3$$

as the first tensorial obstruction to first-order transport and hence as the intrinsic curvature carrier of the connection-first geometry. The natural next question is whether the higher filtration quotients

$$F^m/F^{m+1} \qquad (m \geq 1)$$

also admit an intrinsic geometric interpretation. The answer is affirmative: the full transport filtration is the intrinsic jet hierarchy of the connection-first geometry.

Definition 17.8.1 (Transport jet of order m). Let $p \in ST^{\text{reg}}$, and choose a local flattening gauge near p. The *transport jet of order m* at p is the order–m equivalence class of the local transport law under identification of two gauges when their transport laws agree through order m.

Definition 17.8.2 (Intrinsic jet carrier). For each $m \geq 1$, define the *intrinsic jet carrier of order m* to be the graded transport quotient

$$\mathrm{gr}^m(F) := F^m/F^{m+1}.$$

Lemma 17.8.3 (Order filtration lemma)**.** *Let two local flattening gauges induce transport laws T and T' which agree through order $m-1$. Then their first difference defines a class in*

$$F^m/F^{m+1}.$$

Moreover, this class vanishes if and only if T and T' agree through order m.

Proof. Agreement through order $m-1$ means exactly that all lower-order transport residues have been identified. By definition of the filtration, the first nonvanishing difference then lies in F^m. Two transport laws agree through order m if and only if this difference lies in F^{m+1}. Therefore the first difference is represented canonically by a class in

$$F^m/F^{m+1},$$

and that class vanishes precisely when the two laws agree through order m. □

Lemma 17.8.4 (Gauge-independence of the order–m class)**.** *The class in*

$$F^m/F^{m+1}$$

defined in Theorem 17.8.3 is invariant under change of local flattening gauge modulo lower-order jet data.

Proof. A change of local flattening gauge alters the transport law by a gauge reparametrization. The lower-order terms of this reparametrization are already absorbed by the identification of transport laws modulo the jet data through order $m-1$. The first surviving remainder is therefore well-defined modulo F^{m+1}. Hence the induced class in

$$F^m/F^{m+1}$$

depends only on the order–m transport defect and not on the chosen gauge representative. □

Theorem 17.8.5 (Jet Tower Theorem)**.** *For every integer $m \geq 1$, the graded piece*

$$\mathrm{gr}^m(F) = F^m/F^{m+1}$$

is the intrinsic carrier of the order–m transport jet. More precisely:

(i) *the first-order carrier*

$$\mathrm{gr}^1(F) = F^1/F^2$$

determines the connection class of the first-order transport law;

(ii) *the second-order carrier*

$$\mathrm{gr}^2(F) = F^2/F^3$$

determines the first tensorial obstruction to flatness, namely the curvature carrier;

(iii) *for each $m \geq 3$, the graded piece*

$$\mathrm{gr}^m(F) = F^m/F^{m+1}$$

determines the intrinsic order–m Taylor defect of transport composition, modulo all lower-order transport data.

Equivalently, the filtration

$$F^1 \supseteq F^2 \supseteq F^3 \supseteq \cdots$$

is the intrinsic jet tower of the connection-first geometry.

Proof. We argue by induction on m. For $m = 1$, the first-order transport law is the linear comparison transport

$$\tau_{q \to p} : V_q \to V_p$$

constructed from the morphism locus. By Theorems 17.5.2 and 17.6.4, this is exactly the intrinsic first-order geometric residue of the closed stack. Hence

$$\mathrm{gr}^1(F) = F^1/F^2$$

is the carrier of the first transport jet. For $m = 2$, the statement is precisely Theorem 17.7.1: the first failure of strict first-order transport closure is invisible in degree 1 and appears first in

$$F^2/F^3.$$

This degree–2 carrier is the intrinsic curvature carrier. Assume now that the statement has been proved through order $m - 1$, so that the lower quotients

$$\mathrm{gr}^1(F), \ldots, \mathrm{gr}^{m-1}(F)$$

have already been identified with the intrinsic transport jets through order $m-1$. Choose a local flattening gauge. In that gauge the local transport law admits an expansion whose coefficients through order $m - 1$ are already determined by the lower graded carriers by the induction hypothesis. If two such transport laws agree through order $m - 1$, then by Theorem 17.8.3 their first difference determines a class in

$$F^m/F^{m+1},$$

and this class vanishes exactly when the laws agree through order m. By Theorem 17.8.4, this class is invariant under change of local flattening gauge modulo lower-order jet data. Therefore the order–m defect is intrinsically carried by

$$F^m/F^{m+1}.$$

Hence $\mathrm{gr}^m(F)$ is the intrinsic carrier of the order–m transport jet. This completes the induction. □

Corollary 17.8.6 (Intrinsic Taylor hierarchy)**.** *The local transport law is determined, up to gauge, by the full tower of graded carriers*

$$\{\mathrm{gr}^m(F)\}_{m\geq 1}.$$

In particular, the connection-first geometry admits an intrinsic formal Taylor expansion whose m-th coefficient is carried by

$$F^m/F^{m+1}.$$

Proof. Immediate from Theorem 17.8.5. □

Corollary 17.8.7 (Full differential-geometric reconstruction under convergence)**.** *Assume, in addition, that the intrinsic formal transport expansion defined by the tower*

$$\{\mathrm{gr}^m(F)\}_{m\geq 1}$$

is convergent in local flattening gauges. Then the closed comparison stack reconstructs not merely the connection class and curvature carrier, but the full local differential-geometric transport law.

Proof. By Theorem 17.8.6, the tower

$$\{\mathrm{gr}^m(F)\}_{m\geq 1}$$

determines the formal transport expansion to all orders. If that formal expansion converges, it determines the local transport law itself. Hence the full local differential-geometric structure is reconstructed. □

17.9 Smooth realization

Theorems 13.12.3 and 14.6.9 already prove that smooth geometry is recognized whenever the intrinsic transport algebra admits smooth realization. We now express that fact in the language of connection-first reconstruction.

Theorem 17.9.1 (Smooth realization of the connection-first geometry)**.** *Under the standing principle Standing Principle 1, assume that the intrinsic transport geometry on ST^{reg} admits a compatible smooth realization. Then there exist:*

(i) *a smooth* 4*-dimensional manifold M;*

(ii) *a Lorentz metric g on M, with pointwise signature determined by the realized first-variation pairings;*

(iii) *an affine connection ∇ on TM;*

such that:

(a) *the intrinsic first-order transport class realizes as the connection class of* ∇*;*

(b) *under the inherited second-jet faithfulness condition, nonzero stabilized square classes in* $\mathcal{K}$ *are equivalent to nonzero realized curvature in any compatible smooth realization;*

(c) *under the metric realization already identified in the stack, this curvature agrees with the Riemann curvature tensor of* g*.*

Proof. By hypothesis, the intrinsic transport geometry admits a compatible smooth realization. By Theorem 12.9.1, the intrinsic first-order transport class then realizes as the connection class of an affine connection on the realized regular locus. The pointwise first-variation data already determine the dimension and Lorentzian signature of the realized tangent model on the regular locus. Thus the realization is 4-dimensional and Lorentzian. By Theorem 17.7.1, the stabilized quadratic carrier $\mathcal{K}$ is the intrinsic curvature carrier of the connection-first geometry. The repaired interface statement in Theorem 13.12.3 then yields only the inherited second-jet criterion: under that hypothesis, nonzero stabilized square classes in $\mathcal{K}$ are equivalent to nonzero realized curvature in any compatible smooth realization. This is exactly item (b). In the metric realization already identified in the stack, the realized curvature of that affine connection agrees with the Riemann curvature tensor of the realized Lorentz metric, giving item (c). □

17.10 Interpretation

The chapter may be summarized in one sentence:
In a closed system, intrinsic geometry is connection-first.

The exclusion and the construction are equally rigid. An intrinsic atlas is impossible. Coordinates require representative choice and therefore belong to gauge. The closed-system theorems do not merely fail to reconstruct coordinates; they prove that coordinates are not admissible intrinsic primitives. What is determined is transport. Once the representative sector has been removed, the regular spacetime precursor carries a canonical first-variation family, a canonical connection class, a canonical curvature carrier, and indeed a full intrinsic jet hierarchy. Thus the intrinsic geometric output of the stack is the chain

$$\begin{aligned} ST^{\mathrm{reg}} &\implies \{V_p\}_{p\in ST^{\mathrm{reg}}} \implies \text{intrinsic connection class} \\ &\implies \mathcal{K} \simeq F^2/F^3 \implies \{\mathrm{gr}^m(F)\}_{m\geq 1}. \end{aligned}$$

A coordinate chart appears only after gauge-fixing. The transport class and its full filtration tower exist before that choice.

17.11 Structural synthesis

The geometric endpoint of the closed relational stack is now exact. The stack does not intrinsically reconstruct coordinates, and it does not need to. Coordinates are impossible as canonical objects in a closed comparison system because they depend on representative choice and therefore belong to the noncanonical object locus. What the stack does reconstruct is the intrinsic transport geometry of the regular spacetime precursor. More precisely, it determines:

(i) the canonical spacetime object

$$ST = \varprojlim_{k,t} \big(S_k / F_t^{(k)}\big);$$

(ii) its regular locus ST^{reg};

(iii) the pointwise first-variation arenas

$$V_p \qquad (p \in ST^{\mathrm{reg}}),$$

each of dimension 4;

(iv) the pointwise Lorentzian causal type under the causal-cone hypotheses;

(v) an intrinsic connection class on the first-variation bundle, defined modulo first-order gauge by comparison transport;

(vi) the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3,$$

which is the intrinsic curvature carrier of that geometry;

(vii) the full filtration tower

$$F^1 \supseteq F^2 \supseteq F^3 \supseteq \cdots,$$

whose graded pieces

$$F^m/F^{m+1}$$

are the intrinsic jet carriers of the transport law.

Accordingly, the true geometric spine of the theory is

$$\begin{aligned}(U, \mathcal{C}) &\Longrightarrow \text{closure} \\ &\Longrightarrow \text{quotient semantics} \\ &\Longrightarrow \text{transport} \\ &\Longrightarrow ST \\ &\Longrightarrow \text{connection-first geometry} \\ &\Longrightarrow \mathcal{K} \simeq F^2/F^3 \\ &\Longrightarrow \{\mathrm{gr}^m(F)\}_{m\geq 1}.\end{aligned}$$

Whenever a compatible smooth realization exists, this intrinsic transport geometry realizes as ordinary Lorentzian connection geometry, and the full filtration realizes as the local differential-geometric jet hierarchy. Under the inherited second-jet faithfulness condition, nonzero stabilized square classes in $\mathcal{K}$ are equivalent to nonzero realized curvature in any compatible smooth realization. That is the correct endpoint of the reconstruction program at the present stage of the manuscript.

17.12 The local Einstein arena

Theorems 17.6.4, 17.8.5 and 17.9.1 reconstruct the intrinsic transport geometry of the regular spacetime precursor

$$ST^{\mathrm{reg}}$$

from the closed comparison stack. The reconstruction proceeds without the introduction of coordinates or background manifolds. What is obtained intrinsically is the following hierarchy:

(i) the spacetime precursor

$$ST = \varprojlim_{k,t} \big(S_k / F_t^{(k)}\big);$$

(ii) the regular locus ST^{reg};

(iii) the first–variation arenas

$$V_p \qquad (p \in ST^{\mathrm{reg}}),$$

each a real vector space of dimension 4;

(iv) the intrinsic connection class induced by comparison transport;

(v) the stabilized quadratic carrier

$$\mathcal{K} \simeq F^2/F^3,$$

which is the curvature carrier of this transport geometry.

These objects constitute the full intrinsic geometric output of the closed relational stack prior to any gauge choice. If the transport geometry admits a compatible smooth realization, the connection-first structure reconstructed on ST^{reg} realizes as ordinary Lorentzian geometry. More precisely, there exist

(i) a smooth four–dimensional manifold M,

(ii) a Lorentz metric g on M,

(iii) an affine connection ∇ on TM, such that the intrinsic transport class realizes as the connection class of ∇, and, under the inherited second-jet interface hypothesis, nonzero stabilized square classes in the quadratic carrier are equivalent to nonzero realized curvature for that connection. In particular, the pointwise first–variation arenas V_p realize as the tangent spaces T_pM. Since each V_p is four–dimensional, the realized manifold is four–dimensional as well. Standard differential topology then implies that every point $p \in M$ admits a neighborhood $U \subseteq M$ and a coordinate chart

$$\phi : U \to \mathbb{R}^4$$

which identifies U smoothly with an open subset of $\mathbb{R}^4$. Thus the final geometric output of the reconstruction is the familiar local Einstein arena:

$$M \;\simeq\; \mathbb{R}^4 \qquad \text{locally.}$$

The conceptual direction is therefore inverted relative to the usual presentation of relativistic physics. In the standard formulation of general relativity, one begins by postulating a smooth four–dimensional manifold M equipped with a Lorentz metric g, and the theory proceeds from that geometric background. In the present framework, the manifold is not an assumption. It is the smooth realization of a deeper transport geometry reconstructed from closed comparison structure. Coordinates arise only after a gauge choice, and the local model $\mathbb{R}^4$ appears only at the final stage of the reconstruction. The logical chain is therefore

$$\begin{gathered}(U,\mathcal{C}) \implies \text{closure} \implies \text{transport} \implies ST^{\text{reg}} \\ \implies \text{connection geometry} \implies (M, g, \nabla),\end{gathered}$$

with the classical Einstein starting point emerging as the final local realization

$$M \;\simeq\; \mathbb{R}^4 \quad \text{locally.}$$

The Einstein manifold is thus not the beginning of the theory, but its endpoint.

17.13 Conclusion

The connection-first reconstruction is therefore complete at saved strength: the intrinsic transport geometry is reconstructed unconditionally, while manifold and metric geometry appear as realized endpoints only when a compatible smooth realization exists, not as primitive background assumptions. Accordingly, chapter 18 extends this same intrinsic chain to the electromagnetic phase sector.

Part VI

Electromagnetism

Chapter 18

Electromagnetism from Closedness and Phase Curvature

18.1 Purpose and logical status

Under the standing principle of closed-world admissibility (Standing Principle 1), this chapter develops the phase-curvature consequences of the transport-visibility clause item (SP5) on the stabilized quadratic carrier. As the electromagnetic extension of the main argument, it draws on the transport, quadratic, Hilbert, Einstein-boundary, and connection results of the preceding chapters to derive phase curvature and charge quantization. The purpose of this chapter is to derive the electromagnetic sector determined by the closed comparison framework. The derivation does not proceed in the usual order. Standard gauge theory typically begins with a field, then introduces a bundle, then interprets topological classes, and finally extracts quantization. The closed-system order is the reverse:

$$
\begin{aligned}
&\text{closedness} \Longrightarrow \text{finite admissibility} \Longrightarrow \text{global coherence} \\
&\Longrightarrow \text{discrete phase sectors} \Longrightarrow \text{field laws.}
\end{aligned}
$$

Chapters 6, 7, 12, 13 and 15 already established the structural ingredients needed for this reversal.

1. Every admissible enrichment beyond quotient semantics is exhausted by representative choice and morphism-level transport, possibly with both and with no third locus (Theorems 6.6.1 and 6.6.2).
2. The morphism locus is canonically a flat transport connection on protocol categories, while its later global loop-obstruction description is recorded by loop holonomy (Theorems 5.6.2 and 8.6.1).
3. The first nontrivial finite obstruction to transport is ternary: pairwise compatibility does not yet force global compatibility, but a triangle obstruction does (Theorems 7.5.2 and 9.8.1).

4. The first visible stable obstruction layer is the quadratic carrier

$$\mathcal{K} \cong F^2/F^3.$$

(Theorem 13.10.2)

5. The quadratic carrier admits a canonical orientation involution and a canonical complex structure, unique up to sign (Theorems 15.5.6 and 15.5.8).

6. Under the inherited second-jet faithfulness condition, nonzero stabilized square classes in the degree–2 carrier are equivalent to nonzero realized curvature on the local geometric branch (Theorem 13.12.3).

7. Finite comparison patterns are typed at finite Boolean stage, and global existence is equivalent to coherent extension across the refinement tower (Theorems 1.8.1 and 16.3.3).

The present chapter combines these facts to prove four statements.

1. The morphism locus admits a canonical phase reduction

$$\rho_{\mathrm{ph}} : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to U(1).$$

2. Its smooth realization determines a phase connection

$$A \in \Omega^1(M)$$

with curvature

$$F = dA \in \Omega^2(M).$$

3. The electromagnetic field strength $F = dA$ is the realized phase curvature of the canonical phase reduction, read on the orientation-odd realized degree–2 sector attached to the stabilized quadratic carrier.

4. Charge quantization is determined by closedness, through finite phase typing and coherent tower extension, before any external topological interpretation is introduced.

Thus electromagnetism is not an additional structure placed on top of the closed comparison world. It is the realized phase curvature of the canonical phase reduction of the morphism locus. The chapter therefore establishes:

$$\boxed{\begin{aligned}\text{Electromagnetism} = {} & \text{realized phase curvature} \\ & \text{of the canonical phase reduction of the morphism locus}\end{aligned}}$$

and

$$\boxed{\text{charge quantization} = \text{closedness of the phase sector.}}$$

18.2 The intrinsic loop representation

Let $\mathcal{I}$ be a connected protocol category and fix a base object $i_0 \in \mathrm{Ob}(\mathcal{I})$.

Definition 18.2.1 (Loop group)**.** The loop group of the transport groupoid of $\mathcal{I}$ at i_0 is

$$\Omega_{i_0}(\mathcal{G}(\mathcal{I})) := \mathrm{Aut}_{\mathcal{G}(\mathcal{I})}(i_0).$$

Theorems 6.6.1 and 6.6.2 show that every admissible distinction beyond quotient-level endpoint data is carried by the morphism locus, hence by transport. At the intrinsic level, tree transport is gauge, while loop transport records the later global loop-obstruction description of the morphism locus.

Theorem 18.2.2 (Loop-defect representation)**.** *Under the standing principle Standing Principle 1, there exists a canonical homomorphism*

$$\rho_g : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \longrightarrow H_{\mathrm{stab}}$$

into a stabilizer subgroup H_{stab} such that:

1. *ρ_g is invariant under gauge modification up to conjugation;*
2. *ρ_g is trivial if and only if transport is endpoint-determined;*
3. *all later global loop-obstruction information is carried by ρ_g.*

Proof. By the classification of representative lifts, every admissible lift consists of object representatives together with morphism-level transport data. Changing representatives modifies the transport cocycle by conjugation, so loop holonomy is invariant up to conjugacy. If all loop holonomies are trivial, then transport depends only on endpoints; conversely, any endpoint-determined transport has trivial loop holonomy. Thus the later global loop-obstruction description of the morphism locus is carried by the loop representation ρ_g. □ □

18.3 Triangle obstruction and quadratic localization

The loop representation is not yet the electromagnetic sector. The first task is to identify the precise layer at which intrinsic transport defect first survives.

Definition 18.3.1 (Detection size)**.** Let $\{C_i\}$ be the family of alignment cosets attached to a lifted transport configuration. Its detection size is

$$m := \min\Big\{|I| : \bigcap_{i\in I} C_i = \varnothing\Big\},$$

with $m = \infty$ if no such finite I exists.

Theorem 18.3.2 (Triangle minimality)**.** *The first nontrivial finite obstruction to simultaneous transport compatibility occurs at detection size*

$$m = 3.$$

Equivalently, pairwise compatibility does not yet force trivial loop defect, while a triangle configuration can witness the first genuine intrinsic obstruction.

Proof. By Theorems 7.5.2 and 9.8.1, length–2 transport relations close trivially, whereas length–3 relations produce the first nonvanishing defect. Thus no 2-point observable detects a genuinely new obstruction, but a 3-point observable can. Hence the first finite obstruction is ternary. □ □

The triangle regime is precisely where the first stable obstruction carrier appears.

Corollary 18.3.3 (Quadratic localization)**.** *The intrinsic transport defect factors through the stabilized quadratic carrier*

$$\mathcal{K} \cong F^2/F^3.$$

Proof. The triangle obstruction is first visible at degree 2, and later higher-order terms lie in deeper filtration layers. Therefore the first stable intrinsic carrier of visible transport defect is F^2/F^3. □ □

18.4 Canonical phase reduction

The electromagnetic sector must come from the quadratic carrier, not from an externally imposed gauge group. The crucial point is that the oriented defect double attached to the stabilized quadratic carrier already carries a canonical complex structure.

Theorem 18.4.1 (Canonical phase reduction)**.** *Under the standing principle Standing Principle 1, there exists a canonical abelian phase reduction*

$$\rho_{\mathrm{ph}} : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \longrightarrow U(1),$$

unique up to global phase.

Proof. By Theorem 15.5.6, the oriented quadratic defect sector carries a canonical complex structure

$$J : H_{\mathbb{R}} \to H_{\mathbb{R}}, \qquad J^2 = -\,\mathrm{id},$$

unique up to sign. This equips the oriented quadratic defect sector with a Hermitian structure. The abelianization of the loop-defect representation, composed with the unitary phase determined by that Hermitian structure, yields a phase character

$$\rho_{\mathrm{ph}} : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to U(1).$$

Since the only ambiguity in the underlying complex structure is a global sign, the resulting phase reduction is unique up to global phase. □ □

Remark 18.4.2. The appearance of $U(1)$ is not postulated. It is determined by the canonical complex structure on the oriented defect double attached to the stabilized quadratic carrier.

18.5 Orientation involution and the odd transport sector

The phase reduction alone identifies a unitary sector. To identify electromagnetism itself, one must isolate the orientation-sensitive part of the quadratic carrier.

Definition 18.5.1 (Loop reversal)**.** For a based loop $\gamma \in \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$, define

$$\tau(\gamma) := \gamma^{-1}.$$

Definition 18.5.2 (Orientation involution on the quadratic carrier)**.** The loop-reversal map induces a linear involution

$$\tau_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$$

by functoriality of transport defect under reversal of oriented triangle/loop data.

Lemma 18.5.3. *The induced map satisfies*

$$\tau_{\mathcal{K}}^2 = \mathrm{id}_{\mathcal{K}}\,.$$

Proof. Since $(\gamma^{-1})^{-1} = \gamma$ for every loop γ, the induced action on any functorially attached defect carrier squares to the identity. □ □

Definition 18.5.4 (Even and odd quadratic sectors)**.** Define

$$\mathcal{K}^+ := \ker(\tau_{\mathcal{K}} - \mathrm{id}), \qquad \mathcal{K}^- := \ker(\tau_{\mathcal{K}} + \mathrm{id}).$$

Let H denote the Hilbert completion of the quadratic defect sector constructed from Theorems 15.2.2 and 15.5.6 and its canonical complex structure.

Theorem 18.5.5 (Canonical orientation decomposition)**.** *There is a canonical orthogonal decomposition*

$$H = H^+ \oplus H^-,$$

where $H^\pm$ are the closed ± 1-eigenspaces of the induced orientation involution.

Proof. By Theorem 18.5.3, the induced operator on H is a bounded involution. Therefore its spectrum lies in $\{\pm 1\}$, and H decomposes into the corresponding eigenspaces

$$H = H^+ \oplus H^-.$$

Since the involution is canonical and the Hermitian structure is transport-invariant, this decomposition is canonical and orthogonal. □ □

The odd sector is the one relevant for electromagnetism.

Lemma 18.5.6 (Odd parity determines antisymmetry)**.** *Let $p \in M$ be a realized point and let $k \in \mathcal{K}_p^-$. If*

$$B_k : T_pM \times T_pM \to \mathbb{R}$$

is the realized bilinear representative of k, then

$$B_k(v, w) = -B_k(w, v) \qquad \text{for all } v, w \in T_pM.$$

Proof. Reversal of oriented transport exchanges the ordered pair (v, w) with (w, v). Since k lies in the (-1)-eigenspace of the orientation involution, the realized degree–2 defect changes sign under this exchange. Hence

$$B_k(w, v) = -B_k(v, w),$$

so B_k is alternating. □ □

Corollary 18.5.7 (Degenerate-pair vanishing)**.** *For every $k \in \mathcal{K}_p^-$,*

$$B_k(v, v) = 0 \qquad \text{for all } v \in T_pM.$$

Proof. Apply Theorem 18.5.6 with $w = v$. □ □

Thus every odd quadratic class determines an alternating bilinear form.

Definition 18.5.8 (Odd transport 2-form map)**.** For each realized point $p \in M$, define

$$\Xi_p : \mathcal{K}_p^- \longrightarrow \Lambda^2(T_p^*M)$$

by sending an odd quadratic class to its realized alternating bilinear representative.

Theorem 18.5.9 (Odd sector as 2-form sector)**.** *For each realized point $p \in M$, the odd quadratic sector admits a canonical injective realization*

$$\Xi_p : \mathcal{K}_p^- \hookrightarrow \Lambda^2(T_p^*M).$$

These maps assemble naturally into a bundle morphism

$$\Xi : \mathcal{K}_{\text{real}}^- \to \Lambda^2(T^*M).$$

If the odd quadratic sector exhausts all realized alternating degree–2 transport data, then

$$\mathcal{K}_{\text{real}}^- \cong \Lambda^2(T^*M).$$

Proof. Under the inherited second-jet faithfulness condition, each odd quadratic class on the realized degree–2 channel attached to the stabilized quadratic carrier has a realized bilinear representative on $T_pM \times T_pM$, which provides the existence of Ξ_p. By Theorem 18.5.6 and Theorem 18.5.7, the image lies in $\Lambda^2(T_p^*M)$. Naturality under the realized local symmetry pseudogroup makes these fiberwise maps compatible on overlaps, hence they assemble into a bundle morphism. Injectivity follows because an odd class with vanishing realized alternating representative is invisible in the realized degree–2 branch, hence vanishes in the realized odd sector. If every realized alternating degree–2 transport datum is produced by an odd quadratic class, then the map is also surjective. □ □

Corollary 18.5.10 (Electromagnetic field strength lives in the odd sector)**.** *The electromagnetic field strength constructed below is a section of the odd transport 2-form bundle.*

Proof. By Theorem 18.5.9, the realized odd quadratic sector embeds canonically in $\Lambda^2(T^*M)$. The electromagnetic field strength introduced in Theorem 18.6.3 is the phase curvature and reverses sign under orientation reversal, hence belongs to the odd sector. □

18.6 Phase connection and curvature

The phase reduction determines the electromagnetic connection.

Definition 18.6.1 (Phase line bundle)**.** Let $L_{\text{ph}} \to M$ denote the intrinsic phase line bundle determined by the canonical phase reduction.

Theorem 18.6.2 (Phase connection)**.** *Under smooth realization, the phase reduction determines a local unitary connection*

$$A \in \Omega^1(M).$$

Proof. Theorem 17.9.1 supplies the local transport connection on the realized branch. Passing to the phase reduction of the morphism locus yields a unitary connection on the intrinsic phase line bundle L_{ph}. In local gauge, this is represented by a 1-form A. □ □

Definition 18.6.3 (Electromagnetic field strength)**.** Define

$$F := dA \in \Omega^2(M).$$

Theorem 18.6.4 (Homogeneous Maxwell equation)**.** *The electromagnetic field strength satisfies*

$$dF = 0.$$

Proof. Immediate from Theorem 18.6.3. □ □

Corollary 18.6.5. *The field strength F lies in the odd transport 2-form sector.*

Proof. The phase curvature changes sign under reversal of oriented loop transport, hence lies in the orientation-odd part of the degree–2 carrier. By Theorem 18.5.9, that odd sector is realized as a subbundle of $\Lambda^2(T^*M)$. □ □

18.7 Admissible phase operators and the sourced Maxwell equation

The homogeneous equation is formal. The sourced equation requires the macroscopic naturality mechanism.

Definition 18.7.1 (Admissible phase operator)**.** A phase operator is a map

$$\mathcal{E}_{\mathrm{ph}} : (g, A) \mapsto J_{\mathrm{ph}}$$

from the realized metric and phase connection to a 1-form J_{ph}, satisfying:

1. locality of order at most 1 in A;
2. covariance under the realized local symmetry pseudogroup;
3. gauge equivariance under
$$A \mapsto A + d\chi;$$
4. dependence only on the intrinsic phase sector of the closed system.

Lemma 18.7.2 (Classification of first-order natural phase operators)**.** *Let $F \in \Omega^2(M)$. Any linear, first-order, covariant, natural operator that maps F to a 1-form is, up to overall normalization, given by*

$$F \mapsto d{*}F.$$

Proof. Naturality permits only constructions from g, ∇, and F. A first-order operator producing a 1-form must contract one index of ∇F using the metric. The only such covariant contraction is

$$\nabla^\mu F_{\mu\nu},$$

which in differential-form notation is exactly $d{*}F$, up to normalization. □ □

Definition 18.7.3 (Phase current)**.** Let ψ be a charged matter field in a phase representation sector. The *phase current* J_{ph} is the unique 1-form obtained as the natural variation of the matter coupling with respect to the phase connection A.

Theorem 18.7.4 (Sourced Maxwell equation)**.** *The electromagnetic field satisfies*

$$d{*}F = {*}J_{\mathrm{ph}},$$

where J_{ph} is the phase current associated to charged matter. Equivalently,

$$\nabla_\mu F^{\mu\nu} = g_{\mathrm{ph}}\, J^\nu_{\mathrm{ph}}$$

for a unique coupling normalization g_{ph}.

Proof. By Theorem 18.7.2, the only admissible first-order natural operator on the electromagnetic 2-form is $d{*}F$, up to normalization. The matter sector couples to the phase connection and therefore produces a covariant phase current J_{ph}. Compatibility of the field equation with locality, covariance, gauge equivariance, and first-order dependence determines

$$d{*}F = {*}J_{\mathrm{ph}}$$

up to a single overall constant. Writing that constant in index notation gives

$$\nabla_\mu F^{\mu\nu} = g_{\mathrm{ph}}\, J^\nu_{\mathrm{ph}}.$$

No other first-order, gauge-equivariant, covariant law is admissible. □ □

Corollary 18.7.5 (Charge conservation)**.** *The phase current is conserved:*

$$d{*}J_{\mathrm{ph}} = 0, \qquad \textit{equivalently} \qquad \nabla_\nu J^\nu_{\mathrm{ph}} = 0.$$

Proof. Apply d to both sides of Theorem 18.7.4 and use $d^2 = 0$. □ □

18.8 Finite phase typing

Up to this point the phase sector is present, but it has not yet been shown to be discrete. That discreteness is the real quantization theorem.

Definition 18.8.1 (Finite phase pattern)**.** Let $S \subset \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$ be finite. A finite phase pattern on S is a map

$$\vartheta : S \to U(1).$$

It is called *internally admissible* if it is induced by a finite comparison configuration of the closed system.

Definition 18.8.2 (Stage-k phase sector)**.** For each finite Boolean stage B_k of the refinement tower, let

$$\mathsf{P}_k$$

denote the set of internally admissible phase sectors represented at stage k.

Theorem 18.8.3 (Finite phase typing theorem)**.** *Every internally admissible finite phase pattern induced by the canonical phase reduction*

$$\rho_{\mathrm{ph}} : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to U(1)$$

is represented by a finite-stage Boolean comparison pattern. Equivalently, for every finite

$$S \subset \Omega_{i_0}(\mathcal{G}(\mathcal{I}))$$

and every internally admissible restriction

$$\vartheta = \rho_{\mathrm{ph}}|_S,$$

there exist a finite witness set $F \subset U$, a finite directed comparison pattern $(\pi^{\rightarrow}, \pi^{\leftarrow})$ over F, and an index k such that the realization set

$$X(F; \pi^{\rightarrow}, \pi^{\leftarrow}) \in B_k$$

determines ϑ on S.

Proof. We proceed in three steps. *Step 1: every admissible finite phase restriction is transport data.* By the two-locus theorem, all admissible non-endpoint distinction lies in the morphism locus. Hence any admissible finite phase pattern is determined by transport data on a finite protocol subconfiguration. *Step 2: finite transport data is typed by finite comparison data.* Each loop in a finite set S is represented by a finite zig-zag of morphisms. Therefore the transport information relevant to S depends on only finitely many comparison incidences. By Theorem 16.3.3, this finite transport information is encoded by a finite directed comparison pattern over a finite witness set $F \subset U$. *Step 3: finite comparison data lies at finite Boolean stage.* By Theorem 1.8.1, every finite directed comparison pattern belongs to some finite-stage Boolean algebra B_k. Therefore the phase pattern ϑ is represented at finite Boolean stage. Hence every internally admissible finite phase pattern is finite-stage Boolean. □ □

Corollary 18.8.4 (Discrete finite-stage phase sectors). *For each k, the stage-k phase sector P_k is discrete.*

Proof. Each P_k is represented by admissibility classes inside the finite Boolean algebra B_k. Hence it is combinatorial rather than continuously parametrized. □ □

18.9 Closedness-selected quantization

The phase sector is now finite-stage typed. Closedness upgrades this to global discreteness.

Theorem 18.9.1 (Closedness-selected phase sectors). *Let*

$$\mathsf{P}_\infty$$

denote the set of globally admissible phase sectors. Then restriction along the refinement tower induces a canonical identification

$$\mathsf{P}_\infty \cong \varprojlim_k \mathsf{P}_k.$$

In particular, the set of globally admissible phase sectors is discrete.

Proof. Global admissibility in the closed system is equivalent to existence of a coherent admissible tower. By Theorem 18.8.3, every admissible finite phase restriction lies in some stage-k sector P_k. Hence any globally admissible phase sector determines a coherent tower

$$(\vartheta_k)_k, \qquad \vartheta_k \in \mathsf{P}_k, \qquad \vartheta_{k+1}|_{B_k} = \vartheta_k.$$

Conversely, any such coherent tower defines a global admissible phase sector by the same inverse-limit mechanism that governs all other closed-system objects. Therefore

$$\mathsf{P}_\infty \cong \varprojlim_k \mathsf{P}_k.$$

Since each P_k is discrete by Theorem 18.8.4, the global sector set P_∞ is discrete. □ □

Corollary 18.9.2 (Quantization before geometry)**.** *Charge quantization is a consequence of closedness before smooth geometry is introduced. Under smooth realization, the canonical phase reduction realizes geometrically through the local phase connection A, whose curvature $F = dA$ exhibits the already-discrete phase sector as a quantized phase-curvature sector.*

Proof. By Theorem 18.9.1, the phase sector is already discrete at the internal comparison level. The canonical phase reduction realizes geometrically through the local phase connection A, whose curvature $F = dA$ exhibits that same already-discrete sector on the realized branch. Therefore smooth geometry reflects a quantization already determined by closedness. □ □

Remark 18.9.3 (Logical reversal)*.* The standard order

$$\text{bundle} \Longrightarrow \text{topology} \Longrightarrow \text{quantization}$$

is reversed in the closed-system framework:

$$\text{closedness} \Longrightarrow \text{finite admissibility} \Longrightarrow \text{coherent discrete phase sectors}$$
$$\Longrightarrow \text{quantized phase curvature.}$$

18.10 Charge lattice and the fundamental spectrum

Discreteness of the phase sector now determines quantization of charges.

Definition 18.10.1 (Global phase sector group)**.** Composition of phase sectors induces an abelian group structure on the set P_∞ of globally admissible phase sectors.

Definition 18.10.2 (Primitive phase sector)**.** A nonzero phase sector $\mathfrak{p} \in \mathsf{P}_\infty$ is called *primitive* if whenever

$$\mathfrak{p} = \mathfrak{a}^n$$

for some $\mathfrak{a} \in \mathsf{P}_\infty$ and $n \in \mathbb{Z}$, one has $n = \pm 1$.

Definition 18.10.3 (Charge assignment)**.** A charge assignment is a group homomorphism

$$q : \mathsf{P}_\infty \to \mathbb{R}.$$

Theorem 18.10.4 (Charge lattice theorem)**.** *Assume* P_∞ *is generated by a primitive phase sector* $\mathfrak{p}_0$*. Then every admissible phase sector has the form*

$$\mathfrak{p}_n = \mathfrak{p}_0^n, \qquad n \in \mathbb{Z},$$

and every charge assignment satisfies

$$q(\mathfrak{p}_n) = n\, q_0, \qquad q_0 := q(\mathfrak{p}_0).$$

In particular,

$$q(\mathsf{P}_\infty) = q_0\, \mathbb{Z}.$$

Proof. Every admissible phase sector is a power of the primitive generator $\mathfrak{p}_0$. Since q is a group homomorphism,

$$q(\mathfrak{p}_0^n) = n\, q(\mathfrak{p}_0) = nq_0.$$

Hence the charge image is exactly the rank–1 lattice $q_0\mathbb{Z}$. □ □

Corollary 18.10.5 (Fundamental charge quantization)**.** *All fundamental charges are integer multiples of a primitive charge unit* q_0*.*

Proof. Immediate from Theorem 18.10.4. □

Remark 18.10.6*.* No fundamental fractional charge arises at the primitive phase level. Any effective fractional charge, if present, must therefore be a composite-sector quotient effect rather than a fundamental phase weight.

18.11 Composite-sector descent and the minimal denominator

Section 18.10 closes the fundamental charge lattice. What remains is the emergent possibility of fractional observable charge through quotient semantics at the composite level.

Definition 18.11.1 (Composite charged sector)**.** A composite charged sector is a tensor product of finitely many fundamental charged sectors

$$\mathcal{C} := \mathcal{S}_{n_1} \otimes \cdots \otimes \mathcal{S}_{n_r},$$

with raw charge

$$q_{\text{raw}}(\mathcal{C}) = \Big(\sum_{j=1}^{r} n_j\Big) q_0.$$

Definition 18.11.2 (Observable descent)**.** An observable descent of a composite charged sector is a surjective map

$$\pi_{\text{obs}} : \mathcal{C} \twoheadrightarrow \mathcal{C}_{\text{obs}}$$

such that admissible observables on $\mathcal{C}$ are precisely those that factor through π_{obs}.

Definition 18.11.3 (Descended charge)**.** A descended charge observable on $\mathcal{C}_{\text{obs}}$ is a map

$$q_{\text{obs}} : \mathcal{C}_{\text{obs}} \to \mathbb{R}$$

such that

$$q_{\text{raw}} = m\, q_{\text{obs}} \circ \pi_{\text{obs}}$$

for some positive integer m.

Theorem 18.11.4 (Fractional-charge descent criterion)**.** *A composite observable sector admits an effective fractional charge unit*

$$q_{\text{eff}} = \frac{q_0}{m} \qquad (m > 1)$$

if and only if the raw charge lattice descends through an index-m quotient under observable descent.

Proof. If

$$q_{\text{raw}} = m\, q_{\text{obs}} \circ \pi_{\text{obs}},$$

then a unit observable charge step corresponds to raw charge q_0, hence

$$q_{\text{eff}} = \frac{q_0}{m}.$$

Conversely, if the observable unit is q_0/m, then multiplication by m recovers the raw integer lattice, so the raw charge factors through an index-m quotient. □ □

The transport spine fixes the minimal possible denominator.

Theorem 18.11.5 (Minimal composite quotient index)**.** *If the closed comparison framework determines a nontrivial composite observable descent producing effective fractional charge, then the minimal possible quotient index is*

$$m = 3.$$

Proof. By Theorem 18.11.4, fractional observable charge can arise only through a nontrivial quotient of a composite charged sector. If $m = 2$, the quotient would already be detectable by pairwise compatibility data. But the transport spine proves that no genuinely new obstruction is visible at arity 2; the first irreducible composite obstruction is ternary by Theorem 18.3.2. Therefore the first possible determined quotient must occur at index 3, the first nontrivial composite arity. □ □

18.11.1 Minimal composite descent and uniqueness of denominator 3

Theorem 18.11.6 (Minimal denominator 3 and uniqueness)**.** *Assume a nontrivial composite observable descent exists. Then:*

(i) *the minimal nontrivial denominator is* 3*;*

(ii) *any two minimal denominator-*3 *composite descents are equivalent under admissible quotient descent.*

Proof. **Step 1 (transport visibility: item (SP5)).** Part (i) is exactly Theorem 18.11.5 together with Theorem 18.11.4. The first visible nontrivial composite obstruction is ternary, so the first admissible denominator is forced to be 3.

Step 2 (intrinsic comparison of minimal descents: items (SP1) and (SP5)). For part (ii), let $\pi_{\text{obs}}^{(1)}$ and $\pi_{\text{obs}}^{(2)}$ be two denominator-3 minimal descents. Minimality forces each to be first-visible at the same ternary composite level, so their finite comparison tests agree by Theorem 18.3.2 and the same transport-visibility criterion used in Theorem 18.11.4.

Step 3 (finite-to-global passage and admissible descent: items (SP2) to (SP4)). By comparison completeness of the closed stack, equality on all finite comparison tests implies admissible descent equivalence. Thus the two minimal denominator-3 descents are equivalent. □

Corollary 18.11.7 (Primitive triplet channel)**.** *The minimal denominator-*3 *composite descent determines, up to admissible equivalence, a primitive irreducible internal multiplicity channel of dimension* 3.

Proof. By Theorem 18.11.6, the first nontrivial composite descent has unique index 3. Its observable quotient has exactly three primitive residue classes, and irreducibility follows from minimality (otherwise a proper nontrivial subchannel would descend at smaller index). Therefore the primitive channel is three-dimensional and unique up to admissible equivalence. □

Corollary 18.11.8 (Minimal effective charge unit)**.** *If a nontrivial composite observable descent is determined, then the minimal effective observable charge unit is*

$$q_{\text{eff}} = \frac{q_0}{3}.$$

Proof. By Theorem 18.11.4, effective fractional charge from composite descent has unit q_0/m, where m is the quotient index. By Theorem 18.11.5, the first admissible index is $m = 3$. Hence the minimal effective unit is $q_0/3$. □

Remark 18.11.9. Thus thirds are not imported from phenomenology. They are the denominator determined by the first irreducible composite obstruction size of the transport spine.

18.12 Uniqueness and normalization of the electromagnetic coupling

The electromagnetic field law is already fixed up to a single coupling normalization. We now identify that normalization as internal rather than free.

Definition 18.12.1 (Intrinsic quadratic scale)**.** The intrinsic quadratic scale is the normalization of the canonical quadratic scalar law

$$Q : \mathcal{K} \to \mathbb{R}$$

fixed by the closed-system admissibility structure.

Definition 18.12.2 (Causal-diamond normalization)**.** The causal-diamond normalization is the unique scaling law

$$|D(r)| \sim r^4$$

fixed by the four-dimensional causal-diamond growth theorem.

Theorem 18.12.3 (Uniqueness of the phase-coupling channel)**.** *Every admissible abelian phase-curvature law has the form*

$$\nabla_\mu \mathcal{H}^{\mu\nu} = J^\nu_{\mathrm{ph}}, \qquad \mathcal{H}^{\mu\nu} = c\, F^{\mu\nu},$$

for a unique nonzero constant c. Equivalently, there is exactly one admissible electromagnetic coupling channel, up to overall normalization.

Proof. By Theorem 18.7.2, any admissible first-order, natural, covariant phase law depends on F only through $d{*}F$, up to normalization. Equivalently, any constitutive tensor must be a scalar multiple of F:

$$\mathcal{H}^{\mu\nu} = cF^{\mu\nu}.$$

Thus there is exactly one coupling channel and one remaining scalar normalization. □ □

Theorem 18.12.4 (Intrinsic phase normalization)**.** *The electromagnetic coupling normalization is determined by the same intrinsic normalization mechanism that fixes the quadratic scalar law and the causal-diamond scaling. In particular, the electromagnetic coupling is not an independent free parameter.*

Proof. The first-visible scalar and phase channels inherit their normalization from the stabilized quadratic carrier $\mathcal{K} \cong F^2/F^3$. Under the inherited second-jet faithfulness condition, the Einstein-side scalar coefficient is read on the realized degree–2 channel attached to that stabilized carrier, so the scalar, Einstein, and phase channels are normalized relative to one saved quadratic scale rather than to independent macroscopic inputs. Closedness forbids insertion of an additional independent scale. The causal-diamond growth law fixes the normalization of macroscopic volume and density. Therefore the proportionality constant between phase curvature and phase current is fixed by the unique conversion between the quadratic carrier scale and the causal-diamond density scale. Hence the electromagnetic coupling is determined internally. □ □

Corollary 18.12.5 (Closedness of the electromagnetic sector)**.** *The electromagnetic field equations are fully determined by the closed comparison framework. No free parameter remains in the electromagnetic sector.*

Proof. The field strength is fixed by

$$F = dA,$$

the homogeneous law by

$$dF = 0,$$

the sourced law by

$$d{*}F = {*}J_{\mathrm{ph}},$$

the charge lattice by Theorem 18.10.4, and the coupling normalization by Theorem 18.12.4. Thus the electromagnetic sector is closed. □ □

18.13 Structural summary

The chapter has proved the following chain:

$$\begin{aligned}&\text{morphism locus} \Longrightarrow \text{loop defect} \Longrightarrow \text{triangle obstruction} \Longrightarrow \mathcal{K} \cong F^2/F^3\\ &\Longrightarrow \text{canonical complex phase} \Longrightarrow A \Longrightarrow F = dA,\end{aligned}$$

with

$$F \in \Gamma(\Lambda^-_{\mathrm{tr}}(M)) \subseteq \Omega^2(M),$$

and

$$\begin{cases} dF = 0, \\ d{*}F = {*}J_{\mathrm{ph}}. \end{cases}$$

At the same time, the internal closedness mechanism yields

$$\begin{aligned}&\text{finite phase typing} \Longrightarrow \text{discrete finite-stage phase sectors}\\ &\Longrightarrow \text{coherent global phase lattice}\\ &\Longrightarrow \text{charge quantization.}\end{aligned}$$

Thus the logic of the chapter is not

$$\text{field} \Longrightarrow \text{bundle} \Longrightarrow \text{topology} \Longrightarrow \text{quantization},$$

but rather

$$\begin{aligned}&\text{closedness} \Longrightarrow \text{finite admissibility} \Longrightarrow \text{coherent discrete sectors}\\ &\Longrightarrow \text{electromagnetic field laws.}\end{aligned}$$

This is the precise sense in which the closed-system program inverts the usual order of gauge theory. Accordingly:

Electromagnetism is the realized phase curvature
of the canonical phase reduction of the morphism locus.

Charge quantization is a consequence of closedness.

If effective fractional charge appears, its primitive denominator is 3.

18.14 Uniqueness of the admissible phase operator

The sourced Maxwell equation derived in section 18.7 and Theorem 18.7.4 relied on the classification of admissible phase operators. We now prove that classification in full tensorial form. The result is that there is exactly one admissible first-order operator acting on the electromagnetic field strength.

18.14.1 Statement of the classification theorem

Theorem 18.14.1 (Uniqueness of the admissible phase operator)**.** *Let (M, g) be the realized spacetime manifold with Lorentzian metric, and let*

$$F \in \Omega^2(M)$$

be the abelian phase curvature. Let

$$\mathcal{P} : \Omega^2(M) \longrightarrow \Omega^1(M)$$

be a linear operator satisfying:

1. *locality of differential order at most* 1*;*
2. *covariance under the realized local symmetry pseudogroup;*
3. *tensorial naturality (no dependence on external structure);*
4. *gauge equivariance (dependence only on $F = dA$);*
5. *dependence only on the intrinsic phase sector.*

Then there exists a unique constant $c \in \mathbb{R}$ such that

$$\mathcal{P}(F) = c\, d{*}F.$$

Equivalently, in index notation,

$$\mathcal{P}(F)_\nu = c\, \nabla^\mu F_{\mu\nu}.$$

18.14.2 Reduction to tensorial classification

Lemma 18.14.2 (Gauge reduction). *Any admissible operator $\mathcal{P}$ depends on the connection A only through the curvature $F = dA$.*

Proof. Gauge equivariance requires invariance under

$$A \mapsto A + d\lambda.$$

Since $F = dA$ is gauge-invariant and the theory is abelian, any gauge-equivariant expression must factor through F. Hence $\mathcal{P}$ depends only on F. □ □

Lemma 18.14.3 (First-order form). *Any admissible operator $\mathcal{P}$ is a tensorial contraction of*

$$\nabla_\alpha F_{\beta\gamma}.$$

Proof. By locality of order at most 1, $\mathcal{P}$ may depend on at most one derivative of F. By naturality, the only available structures are g, its inverse, the Levi-Civita connection, and the volume form. Thus $\mathcal{P}(F)$ must be obtained by contracting

$$\nabla_\alpha F_{\beta\gamma}$$

with these structures. □ □

18.14.3 Tensorial classification

Lemma 18.14.4 (Exhaustion of contractions). *All tensorial contractions of $\nabla_\alpha F_{\beta\gamma}$ producing a 1-form are linear combinations of the following two types:*

1. *divergence-type contraction:*
$$\nabla^\mu F_{\mu\nu};$$
2. *antisymmetric 3-form contraction:*
$$\nabla_{[\alpha} F_{\beta\gamma]}.$$

Proof. The tensor $\nabla_\alpha F_{\beta\gamma}$ has one derivative index and two antisymmetric indices. To obtain a 1-form, one must leave exactly one free index and contract the remaining two. *Case 1: contract derivative index with one F index.* This yields

$$\nabla^\mu F_{\mu\nu}.$$

Case 2: antisymmetrize all indices. This yields

$$\nabla_{[\alpha} F_{\beta\gamma]},$$

a 3-form. All other contractions vanish or reduce to these cases:

- contraction of F indices:
$$g^{\beta\gamma}F_{\beta\gamma} = 0$$
by antisymmetry;

- contractions using the Levi-Civita tensor reduce to the Hodge dual of the 3-form above.

Thus the two listed types exhaust all possibilities. □ □

18.14.4 Elimination of the antisymmetric term

Lemma 18.14.5 (Vanishing of the antisymmetric contraction). *The antisymmetric contraction satisfies*
$$\nabla_{[\alpha}F_{\beta\gamma]} = 0.$$

Proof. Since $F = dA$, we have
$$dF = d(dA) = 0.$$

In index notation, this is precisely
$$\nabla_{[\alpha}F_{\beta\gamma]} = 0.$$

□ □

Corollary 18.14.6. *The only nonvanishing admissible contraction is*
$$\nabla^\mu F_{\mu\nu}.$$

Proof. By Theorem 18.14.4, admissible first-order contractions are exhausted by divergence and the fully antisymmetric term. By Theorem 18.14.5, the antisymmetric term vanishes identically. Therefore only $\nabla^\mu F_{\mu\nu}$ remains. □

18.14.5 Normalization

Lemma 18.14.7 (Constancy of the coefficient). *The coefficient multiplying $\nabla^\mu F_{\mu\nu}$ is constant.*

Proof. By naturality, the operator must be invariant under the realized local symmetry pseudogroup. Hence no coordinate-dependent or field-dependent scalar coefficient is admissible. The coefficient must therefore be constant. □ □

18.14.6 Conclusion

Proof of Theorem 18.14.1. By Theorem 18.14.2, $\mathcal{P}$ depends only on F. By Theorem 18.14.3, it is built from ∇F. By Theorem 18.14.4, the only possible contractions are divergence-type or antisymmetric. By Theorem 18.14.5, the antisymmetric term vanishes identically. Thus

$$\mathcal{P}(F)_\nu = c\,\nabla^\mu F_{\mu\nu}.$$

By Theorem 18.14.7, c is constant. In differential-form notation, this is

$$\mathcal{P}(F) = c\,d{*}F.$$

This proves uniqueness. □ □

18.14.7 Maxwell equation as a determined law

Corollary 18.14.8 (Determined form of the phase field equation)**.** *The only admissible first-order, local, covariant, gauge-equivariant field equation for the electromagnetic* 2-*form is*

$$d{*}F = {*}J_{\mathrm{ph}},$$

or equivalently

$$\nabla^\mu F_{\mu\nu} = g_{\mathrm{ph}}\,(J_{\mathrm{ph}})_\nu,$$

for a unique coupling constant g_{ph}.

Proof. By Theorem 18.14.1, the left-hand side must be proportional to $d{*}F$. The right-hand side is the phase current arising from matter coupling. Thus the equation follows uniquely up to normalization. □ □

Remark 18.14.9 (Closure of the electromagnetic sector)*.* This classification removes the final ambiguity in the electromagnetic sector. The field strength F, the homogeneous equation $dF = 0$, the sourced equation $d{*}F = {*}J_{\mathrm{ph}}$, and the charge lattice are all determined by the closed comparison framework. No additional admissible first-order phase law exists.

18.15 Rigidity of the phase symmetry group

Theorems 18.4.1, 18.6.2 and 18.14.1 derive a canonical phase reduction

$$\rho_{\mathrm{ph}} : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to U(1)$$

from the stabilized quadratic carrier

$$\mathcal{K} \cong F^2/F^3.$$

The natural question is whether this $U(1)$ phase symmetry is merely one admissible choice among many, or whether it is the only admissible phase symmetry determined by the closed comparison stack at the first visible transport layer. The answer is rigidity: at degree 2, the phase symmetry is unique.

18.15.1 Admissible phase symmetry groups

Definition 18.15.1 (Admissible phase symmetry group)**.** An *admissible phase symmetry group* at the first visible transport layer is a topological group Γ together with a homomorphism

$$\rho_\Gamma : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to \Gamma$$

satisfying:

1. ρ_Γ depends only on the stabilized quadratic carrier
$$\mathcal{K} \cong F^2/F^3;$$
2. ρ_Γ is natural under gauge modification and under the realized local symmetry pseudogroup;
3. ρ_Γ is induced by a filtration-compatible structure on the quadratic defect sector;
4. the corresponding local connection sector is abelian and first-order, so that its curvature law belongs to the admissible phase class of Chapter 18.

Remark 18.15.2. This definition concerns only *phase symmetry* at the first visible quadratic layer. It does not yet classify all possible symmetry structures that might arise on higher transport-jet layers

$$F^m/F^{m+1}, \qquad m \geq 3.$$

18.15.2 Rigidity at the quadratic layer

Theorem 18.15.3 (Quadratic phase rigidity)**.** *At the first visible transport layer*

$$\mathcal{K} \cong F^2/F^3,$$

the only admissible phase symmetry group is $U(1)$, unique up to global phase. Equivalently, if

$$\rho_\Gamma : \Omega_{i_0}(\mathcal{G}(\mathcal{I})) \to \Gamma$$

is an admissible phase symmetry in the sense of Theorem 18.15.1, then Γ is canonically isomorphic to $U(1)$.

Proof. The proof is by elimination. *Step 1: the phase sector must be induced from the quadratic carrier.* By hypothesis, an admissible phase symmetry depends only on the first visible stabilized transport carrier

$$\mathcal{K} \cong F^2/F^3.$$

No lower layer can contribute, since degree 1 carries only first-order transport data, and no higher layer can contribute to the first visible phase sector without violating the grading of the transport filtration. *Step 2: the only determined extra structure on the quadratic carrier is the canonical complex structure.* Chapter 15 proves that the quadratic carrier carries:

1. a unique positive quadratic scalar form;
2. a unique filtration-compatible orientation involution;
3. a canonical oriented defect double;
4. an orthogonal complex structure

$$J : H_{\mathbb{R}} \to H_{\mathbb{R}}, \qquad J^2 = -\,\mathrm{id},$$

   unique up to sign;
5. a unique Hermitian inner product compatible with J.

Thus the only intrinsic "phase" structure determined on the first visible carrier is a Hermitian complex line structure, unique up to global sign. This is exactly the structure whose unitary symmetry group is $U(1)$. No larger phase group is determined at this layer. □

Continuation of the proof. Step 3: real-linear alternatives are excluded. A merely real-linear phase symmetry would correspond to a structure preserving the positive quadratic form without using the canonical complex structure. But the phase reduction of Chapter 18 is induced precisely by the Hermitian structure determined by J, not just by the underlying real quadratic form. Hence the phase symmetry must be unitary rather than merely orthogonal. In particular, the surviving abelian phase symmetry is not $\mathbb{R}^{\times}$, not $\mathbb{R}$, and not a generic real torus. *Step 4: larger abelian phase groups are excluded.* Because the complex structure on the quadratic defect sector is unique up to sign, there is only one intrinsic phase direction. A larger abelian phase group would require several independent phase directions on the same first visible layer, equivalently several independent compatible complex structures or several independent Hermitian phase sectors. No such multiplicity is proved or allowed by the quadratic Hilbert reconstruction. Therefore the abelian phase symmetry at degree 2 is exactly one copy of $U(1)$. *Step 5: uniqueness up to global phase.* The only residual ambiguity in the complex structure is

$$J \mapsto -J,$$

which corresponds to complex conjugation of the same unitary phase sector. Therefore the resulting phase reduction is unique up to global phase. Hence every admissible phase symmetry group at the first visible quadratic layer is canonically isomorphic to $U(1)$. □ □

18.15.3 Exclusion of higher-layer phase extensions

Theorem 18.15.4 (No independent higher-layer phase extension at degree 2)**.** *Let*

$$m(F) = F^m/F^{m+1}$$

be the higher transport-jet carriers of Chapter 17. Then no admissible phase symmetry at the first visible electromagnetic layer can depend on any $m(F)$ with $m \geq 3$. Equivalently, the electromagnetic phase symmetry is already closed on

$$F^2/F^3.$$

Proof. Chapter 17 proves that for $m \geq 3$, the graded pieces

$$F^m/F^{m+1}$$

are higher-order transport jets: they encode order–m Taylor defect modulo lower-order data. They therefore belong to higher transport structure, not to the first visible phase layer. An admissible phase symmetry at the electromagnetic level must belong to the first visible obstruction layer itself. If it depended essentially on some F^m/F^{m+1} with $m \geq 3$, then the phase sector would not be first-visible at degree 2, contrary to the construction of the chapter and to the localization

$$\mathcal{K} \cong F^2/F^3.$$

Hence the electromagnetic phase symmetry is already exhausted on the quadratic layer and admits no independent extension from higher transport-jet carriers. □ □

18.15.4 Consequences

Corollary 18.15.5 (Uniqueness of electromagnetism at the quadratic layer)**.** *At the first visible transport layer, electromagnetism is the unique admissible phase gauge theory.*

Proof. By Theorem 18.15.3, the only admissible phase symmetry group is $U(1)$. By Theorem 18.14.1, the unique admissible first-order field law on that $U(1)$-sector is the Maxwell law

$$dF = 0, \qquad d{*}F = {*}J_{\mathrm{ph}}.$$

Therefore electromagnetism is the unique admissible phase gauge theory at degree 2. □ □

Remark 18.15.6 (What remains open). The present theorem does *not* prove that no further non-abelian symmetry can arise anywhere in the full transport stack. It proves the narrower and sharper claim:

at the first visible quadratic layer, the only admissible phase symmetry is $U(1)$.

Any additional gauge structure, if it exists, must therefore arise from a genuinely higher transport layer rather than from the electromagnetic phase sector itself.

18.16 Conclusion

At the first visible quadratic layer, the electromagnetic conclusion is rigid: phase symmetry is canonically $U(1)$, admissible first-order dynamics is Maxwell, and charge quantization is fixed by the same closure constraints. No independent degree–2 alternative remains.

This is the forced handoff point of the gauge arc: once the degree–2 phase sector is closed, any additional internal or matter structure must come from higher transport forcing rather than from modifying electromagnetism itself; that continuation is carried by chapter 19.

Accordingly, the scalar channel fixed by the main stack is carried next through matter-sector classification, numerical closure, and canonical realization in chapters 19 to 22, before Appendix chapter B records its downstream magnetic-confinement application, isolating the canonical intrinsic transport scalar and separating later perturbative and confinement-level readings.

Chapter 19

Intrinsic Scalar Invariant and Discrete Hierarchy of Matter Sectors

19.1 Introduction and logical status

The theorem package (Theorems 19.2.7 and 19.2.8) establishes that every primitive excitation sector of the closed comparison system $(U, \mathcal{C})$ is classified, up to canonical unitary equivalence, by irreducible representation data of the internal gauge product

$$G_{\text{int}} := U(1) \times SU(2) \times SU(3).$$

The remaining classification problem is to distinguish sectors that carry identical gauge representation data. The present chapter constructs such a distinguishing invariant intrinsically from the stabilized quadratic transport carrier

$$\mathcal{K} \cong F^2/F^3.$$

Specifically, the chapter proves three things. First, under the inherited second-jet faithfulness condition, the restriction of the canonical scalar channel to the integral carrier is identified, on the realized degree–2 channel attached to the stabilized quadratic carrier, with the scalar curvature projection:

$$Q|_{\mathcal{K}} = c \cdot \mathrm{Scal}(\cdot), \qquad c > 0.$$

Second, the resulting scalar invariant $m_0(\sigma) := Q(\kappa_\sigma)$ is strictly positive and transport-invariant. Third, the refinement depth k_σ separates all primitive sectors within a fixed gauge class, yielding a discrete hierarchy indexed by $\mathbb{N}$.

The logical dependencies are as follows.

(i) Stabilization of the quadratic carrier and interface detection: Theorems 13.10.2 and 13.10.4.

(ii) Under the inherited second-jet faithfulness condition, nonzero stabilized square classes in the stabilized quadratic carrier are equivalent to nonzero realized curvature in any compatible smooth realization, and the scalar-channel argument below uses the realized degree–2 channel attached to that carrier: Theorems B.4.2 and 13.12.3.

(iii) Isotropy completeness: Theorem 16.6.5.

(iv) The admissible branch weight is additive on $\mathcal{K}$, $O(V, g_p)$-invariant, nonnegative on admissible classes, and the admissible scalarization space is one-dimensional: Theorem 14.7.2 condition (S4), Theorems 14.7.4 and 14.7.5.

(v) The scalar summand of the curvature module is one-dimensional; the Weyl and trace-free Ricci summands carry no $O(V, g_p)$-invariant linear functional: section 16.9 and Theorem 16.9.5.

(vi) Canonical quadratic defect section, its basepoint-independence, and functorial uniqueness: Theorem B.4.4.

(vii) Existence, strict positivity, and quadratic extension of the scalar channel: Theorems 14.8.2, 15.2.2 and 15.2.4.

(viii) Finite typing, compactness, and stability: Theorems 1.8.1 and 18.8.3.

(ix) Inverse-limit separation: Theorem 1.7.1.

(x) Classification and finiteness of primitive transport mechanisms: Theorems 19.2.4 and 19.2.5.

(xi) Primitive excitation dichotomy: Theorem 19.2.6.

(xii) No-third-locus exhaustion of representative enrichment: Theorem 5.1.1.

No additional axiom, structural hypothesis, or dynamical assumption is introduced beyond what is already established in Chapters 1–19.

19.2 Primitive mechanism package and spectrum reduction

Definition 19.2.1 (Primitive excitation mechanism)**.** A *primitive excitation mechanism* is a nontrivial transport-visible sector mechanism that is irreducible at its first finite visible stage, in the sense that it is not an admissible composite of lower-stage mechanisms of the same locus type.

Lemma 19.2.2 (Object-locus reduction)**.** *Up to admissible equivalence, every primitive object-locus excitation mechanism reduces to the canonical orientation-splitting doublet channel.*

Proof. Let $\mathfrak{m}$ be a primitive object-locus excitation mechanism. By the lift classification (Theorems 6.6.1 and 6.6.2), object-locus data are exactly representative-choice data. At the primitive level, the only nontrivial object-locus first-visible transport distinction is orientation splitting on the quadratic carrier. Its existence and uniqueness up to admissible filtration-compatible equivalence are given by Theorem 15.3.5. Moreover, Theorem 15.3.6 identifies the corresponding primitive irreducible block as a two-dimensional doublet. Since $\mathfrak{m}$ is primitive (Theorem 19.2.1), it must coincide with this minimal object-locus primitive channel up to admissible equivalence. Therefore every primitive object-locus mechanism reduces to the canonical orientation-splitting doublet channel. □

Lemma 19.2.3 (Morphism-locus reduction)**.** *Up to admissible equivalence, every primitive morphism-locus excitation mechanism reduces to minimal composite descent of index* 3*, i.e. to the primitive triplet channel.*

Proof. By Theorems 18.11.5 and 18.11.6, the first nontrivial primitive morphism-locus descent has unique minimal index 3. Then Theorem 18.11.7 identifies the associated primitive irreducible channel as a triplet, unique up to admissible equivalence. □

Theorem 19.2.4 (Primitive mechanism uniqueness)**.** *Primitive excitation mechanisms are exhausted, up to admissible equivalence, by the following two primitive classes, possibly with both and with no third primitive enrichment class:*

(i) *object-locus orientation splitting (doublet channel);*

(ii) *morphism-locus minimal composite descent (triplet channel).*

Proof. **Step 1 (exhaustion of possible loci).** By the lift classification (Theorems 6.6.1 and 6.6.2), primitive mechanism data can occur only in representative-choice (object-locus) or transport (morphism-locus) coordinates. By Theorem 5.1.1, there is no third independent enrichment locus. Hence every primitive excitation mechanism belongs to one of these two locus types.

Step 2 (uniqueness inside each locus). If a primitive mechanism is object-locus, then Theorem 19.2.2 reduces it, up to admissible equivalence, to the canonical orientation-splitting doublet channel. If a primitive mechanism is morphism-locus, then Theorem 19.2.3 reduces it, up to admissible equivalence, to minimal composite descent of index 3, i.e. the primitive triplet channel. So each of the two loci contributes exactly one primitive type up to admissible equivalence.

Step 3 (exhaustion of primitive types). Step 1 shows that every primitive mechanism is exhausted by the object and morphism locus forms with no third primitive enrichment class, while Step 2 shows that each of those two classes has the listed canonical primitive realization up to admissible equivalence. Therefore primitive excitation mechanisms are exhausted by the two listed classes: object-locus orientation splitting and morphism-locus minimal composite descent. □

Theorem 19.2.5 (No primitive proliferation)**.** *No primitive excitation mechanism exists beyond the two classes of Theorem 19.2.4.*

Proof. By Theorem 5.1.1, representative enrichment is exhausted by object and morphism loci with no third source. By Theorem 19.2.4, each locus has a unique primitive mechanism type up to admissible equivalence. Therefore no additional primitive mechanism can appear. □

Theorem 19.2.6 (Primitive excitation two-class exhaustion). *Every nontrivial primitive excitation sector is exhausted by the two primitive classes*

(i) *object-locus orientation-splitting class;*

(ii) *morphism-locus minimal composite-descent class.*

A sector may use either of these loci, or both, but no third primitive enrichment class occurs.

Proof. By Theorems 6.6.1 and 6.6.2, every transport-visible primitive sector is carried by object-locus data, morphism-locus data, or both. By Theorem 5.1.1, there is no third independent enrichment locus. By Theorem 19.2.4, the object locus contributes only the canonical orientation-splitting doublet channel and the morphism locus contributes only the primitive minimal composite-descent triplet channel, up to admissible equivalence. By Theorem 19.2.5, no additional primitive mechanism exists beyond those two locus-wise primitive classes. Hence every nontrivial primitive excitation sector is exhausted by the two listed primitive classes, with no third primitive enrichment class. □

Theorem 19.2.7 (Matter spectrum reduction). *Under Standing Principle 1, primitive sector classification reduces to irreducible representation data of*

$$G_{\text{int}} := U(1) \times SU(2) \times SU(3),$$

with factors fixed intrinsically by:

(i) *canonical phase rigidity for the abelian factor (Theorem 18.15.3);*

(ii) *primitive object-locus doublet channel (Theorem 15.3.6);*

(iii) *primitive morphism-locus triplet channel (Theorem 18.11.7).*

Proof. The phase factor is fixed by Theorem 18.15.3, giving the canonical one-dimensional abelian unitary channel. By Theorems 19.2.2 and 19.2.3, the only primitive non-abelian multiplicity channels are the doublet and triplet channels. By Theorem 19.2.5, no additional primitive channel is available. Therefore the primitive sector spectrum is exactly reduced to representation data of the stated product. □

Corollary 19.2.8 (Unique forced primitive output). *The primitive internal output is uniquely forced, up to admissible equivalence, to the* 1-, 2-, *and* 3-*channel package encoded by* $U(1) \times SU(2) \times SU(3)$.

Proof. By Theorem 19.2.7, every primitive sector is classified, up to admissible equivalence, by irreducible representation data of

$$U(1) \times SU(2) \times SU(3),$$

with the three factors fixed intrinsically by phase rigidity, primitive object-locus splitting, and primitive morphism-locus descent. By Theorem 19.2.5, no additional primitive mechanism exists beyond these forced channels. Hence the primitive internal output is uniquely determined, up to admissible equivalence, by the 1-, 2-, and 3-channel package. □

Remark 19.2.9 (Role of the no-third-locus theorem). The separation argument requires that every transport-distinguishing datum between sectors factors through the two classified primitive mechanisms. By Theorem 5.1.1, every representative enrichment of quotient semantics is exhausted by the object locus and the morphism locus with no third independent source. Under Theorems 19.2.4 and 19.2.5, every such mechanism is enumerated and fully encoded at its minimal stage. Hence any inter-sector distinction must be of a type already classified and visible at the minimal stage.

19.3 Orientation involution on the stabilized carrier

The stabilized quadratic carrier inherits a canonical involution from loop reversal, established in sections 13.11 and 18.5.

Definition 19.3.1 (Orientation involution)**.** The loop-reversal map $\tau(\gamma) := \gamma^{-1}$ on transport loops induces, by the loop-reversal construction of section 18.5, a well-defined linear involution

$$\tau_{\mathcal{K}} : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}, \qquad \tau_{\mathcal{K}}^2 = \mathrm{id},$$

on $\mathcal{K}_{\mathbb{R}} := \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 19.3.2 (Canonical eigenspace decomposition)**.** *There is a canonical orthogonal direct-sum decomposition*

$$\mathcal{K}_{\mathbb{R}} = \mathcal{K}^{+} \oplus \mathcal{K}^{-}, \qquad \mathcal{K}^{\pm} := \ker(\tau_{\mathcal{K}} \mp \mathrm{id}),$$

into the ± 1 *eigenspaces of* $\tau_{\mathcal{K}}$*. This coincides with the decomposition* $H = H^{+} \oplus H^{-}$ *of section 18.5.*

Proof. Since $\tau_{\mathcal{K}}^2 = \mathrm{id}$, the operator $\tau_{\mathcal{K}}$ is an orthogonal involution on $(\mathcal{K}_{\mathbb{R}}, g_{\mathcal{K}})$ (Theorem 15.2.6) with minimal polynomial dividing $(x-1)(x+1)$. The standard spectral decomposition yields the stated splitting; orthogonality follows because $\tau_{\mathcal{K}}$ is orthogonal. The identification with $H^{\pm}$ is section 18.5. □

Definition 19.3.3 (Orientation parity)**.** For a nonzero class $x \in \mathcal{K}_{\mathbb{R}}$, the *orientation parity* is

$$\epsilon(x) := \begin{cases} +1 & x \in \mathcal{K}^{+}, \\ -1 & x \in \mathcal{K}^{-}. \end{cases}$$

Remark 19.3.4 (Parity is a $\mathbb{Z}_2$-grading). Orientation parity records the bosonic/fermionic split of Theorem 19.2.6; it does not separate sectors within a single parity class. The hierarchy invariant below is strictly finer.

19.4 Defect class of an excitation sector

Definition 19.4.1 (Defect class)**.** Let σ be a primitive excitation sector (Theorem 19.2.1). Fix a representative state $u_\sigma \in U$. The *defect class* of σ is

$$\kappa_\sigma := \kappa(u_\sigma) \in \mathcal{K},$$

where $\kappa : U \to \mathcal{K}$ is the canonical quadratic defect section of Theorem B.4.4.

Lemma 19.4.2 (Well-definedness)**.** *The class κ_σ is independent of the choice of representative.*

Proof. If u_σ, u'_σ both represent σ, there exists $g \in \mathrm{Aut}(U, \mathcal{C})$ with $u'_\sigma = g \cdot u_\sigma$. By equivariance of κ (Theorem B.4.4), $\kappa(u'_\sigma) = g_*\kappa(u_\sigma)$. By the functorial uniqueness clause of Theorem B.4.4, g_* preserves the class in $\mathcal{K}$, so $\kappa(u'_\sigma) = \kappa(u_\sigma)$. □

Lemma 19.4.3 (Nontriviality implies nonzero defect class)**.** *If σ is a nontrivial primitive sector, then $\kappa_\sigma \neq 0$.*

Proof. A nontrivial sector produces a nontrivial transport perturbation. By Theorems 13.10.2 and 13.10.4, every nontrivial transport obstruction has a nonzero representative in $\mathcal{K}$. Hence $\kappa_\sigma \neq 0$. □

19.5 Identification of the scalar channel

Under the inherited second-jet faithfulness condition, this section identifies the restriction $Q|_{\mathcal{K}}$ on the realized degree–2 channel attached to the stabilized quadratic carrier with the scalar curvature projection of the defect class.

Remark 19.5.1 (Two levels of the scalar channel). The canonical scalar channel operates at two levels. At the *integral level*, $Q|_{\mathcal{K}} : \mathcal{K} \to \mathbb{R}_{\geq 0}$ is a nonnegative additive map on the abelian group $\mathcal{K}$; nonnegativity holds on admissible defect classes (Theorem 14.7.2). At the *real level*, $Q_{\mathbb{R}} : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ is the unique positive quadratic extension (Theorem 15.2.2). Theorem 19.5.3 below concerns the integral level; the form of $Q_{\mathbb{R}}$ is an open problem (Theorem 19.5.4).

Remark 19.5.2 (Nonnegativity of scalar curvature on admissible classes). The scalar curvature $\mathrm{Scal} : \mathcal{R}(V) \to \mathbb{R}$ is sign-indefinite on the full curvature module. Under the inherited second-jet faithfulness condition, however, admissible defect classes $\kappa \in \mathcal{K}$ are read on the realized degree–2 channel attached to the stabilized quadratic carrier, and on that channel Theorem 19.5.3 gives

$$Q(\kappa) = c \cdot \mathrm{Scal}(\kappa), \qquad c > 0.$$

Since the admissible branch weight is nonnegative on admissible defect classes (Theorem 14.7.2), it follows under that inherited hypothesis that

$$\mathrm{Scal}(\kappa) \geq 0 \qquad \text{for admissible } \kappa \in \mathcal{K}.$$

This is not an additional axiom: it records the sign consequence of the repaired scalar-channel identification on admissible classes, not a positivity claim on the full curvature module.

Theorem 19.5.3 (Scalar channel identification)**.** *Under the inherited second-jet faithfulness condition, the canonical scalar channel $Q : \mathcal{K} \to \mathbb{R}_{\geq 0}$ satisfies*

$$Q(\kappa) = c \cdot \mathrm{Scal}(\kappa) \qquad (\kappa \in \mathcal{K}),$$

for a universal constant $c > 0$, where

$$\mathrm{Scal}(\kappa) := \Phi \circ \mathrm{pr}_{\mathrm{scal}} \circ \mathsf{Jet}_2(\kappa),$$

$\mathsf{Jet}_2 : \mathcal{K} \to \mathcal{R}(V)$ *is used only through the realized degree–2 channel attached to the stabilized quadratic carrier under that inherited faithfulness condition,* $\mathrm{pr}_{\mathrm{scal}} : \mathcal{R}(V) \to \mathcal{S}(V)$ *is the $O(V, g_p)$-equivariant projection onto the one-dimensional scalar summand (section 16.9), and $\Phi : \mathcal{S}(V) \to \mathbb{R}$ is a fixed nonzero $O(V, g_p)$-invariant linear functional.*

Proof. **Step 1: $Q|_{\mathcal{K}}$ is additive and $O(V, g_p)$-invariant.** By Theorem 14.7.5, Q is a positive multiple of every admissible branch weight. By condition (S4) (Theorem 14.7.2), every admissible branch weight is additive on $\mathcal{K}$ as a group homomorphism and natural under the realized symmetry flow. Hence $Q|_{\mathcal{K}}$ is additive on $\mathcal{K}$ and invariant under the realized local symmetry group. By Theorem 16.6.5, the realized local symmetry group at each point p is $O(V, g_p)$. Therefore, under the inherited second-jet faithfulness condition, the transported scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier is an $O(V, g_p)$-invariant nonnegative additive map.

Step 2: Passage from the stabilized quadratic carrier to its realized degree–2 channel. By Theorems B.4.2 and 13.12.3, the map $\mathsf{Jet}_2 : \mathcal{K} \to \mathcal{R}(V)$ carries the stabilized quadratic carrier into the realized degree–2 channel attached to that carrier, where nonzero stabilized square classes are equivalent to nonzero realized curvature.

Step 3: $O(V, g_p)$-invariant additive maps on $\mathcal{R}(V)$ span a one-dimensional space. The $O(V, g_p)$-irreducible decomposition

$$\mathcal{R}(V) = \mathcal{W}(V) \oplus \mathcal{E}(V) \oplus \mathcal{S}(V)$$

is established in section 16.9; $\mathcal{S}(V)$ is one-dimensional (Theorem 16.9.5). Since $\mathcal{W}(V)$ and $\mathcal{E}(V)$ are non-trivial irreducible $O(V, g_p)$-representations,

$$\mathrm{Hom}_{O(V,g_p)}\big(\mathcal{W}(V), \mathbb{R}\big) = 0, \qquad \mathrm{Hom}_{O(V,g_p)}\big(\mathcal{E}(V), \mathbb{R}\big) = 0.$$

Every $O(V, g_p)$-invariant additive map $\mathcal{R}(V) \to \mathbb{R}$ therefore annihilates $\mathcal{W}(V) \oplus \mathcal{E}(V)$ and factors through $\mathrm{pr}_{\mathrm{scal}}$. The space of such maps is one-dimensional, spanned by $\Phi \circ \mathrm{pr}_{\mathrm{scal}}$.

Step 4: Conclusion. By Steps 1–3, that transported scalar channel on the realized degree–2 channel is $c' \cdot (\Phi \circ \mathrm{pr}_{\mathrm{scal}})$ for some $c' \in \mathbb{R}$. Since Q is nonnegative and nontrivial on admissible alternatives (Theorem 15.2.4), we have $c' > 0$. Set $c := c'$; pulling this identity back along that realized degree–2 channel gives the stated formula. □

Remark 19.5.4 (Scope of the identification). Theorem 19.5.3 identifies $Q|_{\mathcal{K}}$ as an additive map on the integral carrier $\mathcal{K}$. The symmetric bilinear source of the quadratic extension is forced by polarization (Theorem 14.8.2). What remains open here is not the existence of a bilinear source, but the fully explicit real-level formula relating $Q_{\mathbb{R}} : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ to Scal beyond the integral carrier. Determining that explicit formula for $Q_{\mathbb{R}}$ remains an open problem.

Remark 19.5.5 (Relation to the cosmological scalar). Under the inherited second-jet faithfulness condition, the map $Q|_{\mathcal{K}} = c \cdot \mathrm{Scal}(\cdot)$ is the same scalar projection as the cosmological coefficient $\beta_{\triangle}$ in Theorem 16.9.5. Consequently, under that inherited hypothesis, the Born probability law (Theorem 14.7.6) and the cosmological scalar term arise from the same scalar projection on the realized degree–2 channel attached to the stabilized quadratic carrier.

19.6 The canonical scalar invariant

Definition 19.6.1 (Scalar invariant)**.** For a primitive excitation sector σ, the *scalar invariant* is

$$m_0(\sigma) := Q(\kappa_\sigma),$$

where $Q : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ is the canonical quadratic scalar channel of Theorem 15.2.2. Under the inherited second-jet faithfulness condition, on the integral carrier $m_0(\sigma) = c \cdot \mathrm{Scal}(\kappa_\sigma)$ by Theorem 19.5.3.

Theorem 19.6.2 (Strict positivity)**.** *For every nontrivial primitive sector σ, $m_0(\sigma) > 0$.*

Proof. By Theorem 19.4.3, $\kappa_\sigma \neq 0$. By Theorem 15.2.4, $Q(x) > 0$ for every nonzero $x \in \mathcal{K}_{\mathbb{R}}$. Hence $m_0(\sigma) = Q(\kappa_\sigma) > 0$. □

Theorem 19.6.3 (Transport invariance)**.** *The scalar $m_0(\sigma)$ is invariant under admissible transport.*

Proof. Let $\Phi_t \in \mathrm{Aut}(U, \mathcal{C})$ and $u \in U$. By equivariance of the canonical quadratic defect section (Theorem B.4.4),

$$\kappa(\Phi_t(u)) = \Phi_{t*}\kappa(u).$$

By Theorem 15.2.2, the restriction of Q to the integral carrier is the canonical degree–2 scalar channel, and by the naturality clause in condition (S4) of Theorem 14.7.2 that scalar channel is preserved by admissible symmetry flow on the realized degree–2 channel attached to the stabilized quadratic carrier. Therefore

$$Q(\Phi_{t*}k) = Q(k) \qquad (k \in \mathcal{K}).$$

Applying this to $k = \kappa(u)$ gives

$$Q(\kappa(\Phi_t(u))) = Q(\Phi_{t*}\kappa(u)) = Q(\kappa(u)).$$

Hence $m_0(\sigma) = Q(\kappa_\sigma)$ is preserved under admissible transport. ☐

Remark 19.6.4 (Non-injectivity of m_0)*.* Under the inherited second-jet faithfulness condition, if two distinct sectors have the same scalar-curvature coordinate of their defect class, then they share the same value of m_0. The separation theorem uses the strictly finer invariant k_σ.

19.7 Refinement depth

The stabilized carrier is $\mathcal{K} = \varprojlim_k \mathcal{K}_k$ (Theorem 13.10.2), with projections $\pi_k : \mathcal{K} \to \mathcal{K}_k$.

Definition 19.7.1 (Refinement depth)**.** For a nontrivial primitive sector σ, the *refinement depth* is

$$k_\sigma := \min\big\{k \in \mathbb{N} : \pi_k(\kappa_\sigma) \neq 0\big\}.$$

Proposition 19.7.2 (Finiteness of refinement depth)**.** *Every nontrivial primitive sector has $k_\sigma < \infty$.*

Proof. By Theorem 19.4.3, $\kappa_\sigma \neq 0$ in $\varprojlim_k \mathcal{K}_k$. By Theorem 1.7.1, at least one stage projection is nonzero, so the minimum exists in $\mathbb{N}$. ☐

Proposition 19.7.3 (Transport invariance)**.** *The refinement depth k_σ is invariant under admissible transport.*

Proof. Automorphisms of $(U, \mathcal{C})$ preserve the projective system $\{\mathcal{K}_k\}$ and commute with the projections π_k by functoriality of the defect construction (Theorem B.4.4). Hence the minimal nonzero stage index is preserved. ☐

19.8 Finite-stage determination and stage rigidity

Lemma 19.8.1 (Finite-stage separation)**.** *Let σ_1, σ_2 be distinct primitive sectors. Then there exists $k \in \mathbb{N}$ with $\pi_k(\kappa_{\sigma_1}) \neq \pi_k(\kappa_{\sigma_2})$.*

Proof. By Theorem 1.7.1, elements of $\varprojlim_k \mathcal{K}_k$ are equal if and only if all stage projections agree. Contrapositive: $\kappa_{\sigma_1} \neq \kappa_{\sigma_2}$ implies the existence of such k.

We show $\sigma_1 \neq \sigma_2$ forces $\kappa_{\sigma_1} \neq \kappa_{\sigma_2}$. By Theorem 6.6.1, all transport distinction is carried by the morphism locus. By Theorem 18.8.3, any such distinction is typed by finite comparison data in some stage-k Boolean algebra $\mathcal{B}_k$. If two sectors agreed at every finite stage they would agree in the inverse limit by Theorems 1.8.1 and 18.9.1, contradicting inequivalence. ☐

Lemma 19.8.2 (Stage rigidity)**.** *Let σ_1, σ_2 be distinct primitive sectors with identical gauge data and $k_{\sigma_1} = k_{\sigma_2} = k$. If $\pi_k(\kappa_{\sigma_1}) = \pi_k(\kappa_{\sigma_2})$, then $\sigma_1 = \sigma_2$.*

Proof. Assume $\pi_k(\kappa_{\sigma_1}) = \pi_k(\kappa_{\sigma_2})$. Both defect classes vanish below stage k by minimality and coincide at stage k by assumption. Suppose for contradiction that $\sigma_1 \neq \sigma_2$. By Theorem 19.8.1, there exists $k' > k$ at which the projections differ; new distinguishing data must appear above stage k.

Step 1. By Theorem 5.1.1, every representative enrichment is exhausted by the object locus and the morphism locus. Any inter-sector distinction must lie in one of these two channels.

Step 2. By Theorem 19.2.4, the primitive mechanisms are exactly object-locus orientation splitting (Theorem 19.2.2) and morphism-locus minimal composite descent of index 3 (Theorem 19.2.3). By Theorem 19.2.5, no further primitive mechanism exists. Each is characterized up to admissible equivalence by Theorems 15.3.5 and 18.11.6 and is fully encoded at its minimal stage: orientation splitting by the stage-k class of Theorem 15.3.6; index-3 descent by the stage-k class of Theorem 18.11.7.

Step 3. A new distinguishing datum at stage $k' > k$ would require either a new primitive mechanism (excluded by Theorem 19.2.5) or a new enrichment locus (excluded by Theorem 5.1.1). Hence all finite stages agree, so $\sigma_1 = \sigma_2$ by Theorems 1.8.1 and 18.9.1, a contradiction. □

19.9 Separation of sectors with identical gauge labels

Theorem 19.9.1 (Separation by refinement depth)**.** *Let σ_1, σ_2 be nontrivial primitive sectors carrying identical gauge representation data for $U(1)\times SU(2)\times SU(3)$. If $\sigma_1 \neq \sigma_2$, then $k_{\sigma_1} \neq k_{\sigma_2}$.*

Proof. Suppose for contradiction that $k_{\sigma_1} = k_{\sigma_2} = k$.

Case 1: $\pi_k(\kappa_{\sigma_1}) \neq \pi_k(\kappa_{\sigma_2})$. By Theorem 19.2.6, the locus type of a sector is determined by its gauge labels. By Theorem 19.2.4, at most one primitive mechanism exists per locus type within a fixed gauge class. Two distinct sectors sharing gauge labels, locus type, and depth k would require two primitive mechanisms of the same locus type in the same gauge class, contradicting uniqueness.

Case 2: $\pi_k(\kappa_{\sigma_1}) = \pi_k(\kappa_{\sigma_2})$. By Theorem 19.8.2, $\sigma_1 = \sigma_2$, contradicting the assumption.

Both cases yield contradictions. □

Corollary 19.9.2 (Discrete hierarchy within each gauge class)**.** *Within any fixed gauge representation class, the assignment $\sigma \mapsto k_\sigma$ is injective on primitive sectors; in particular, they are indexed by a discrete subset of $\mathbb{N}$.*

Proof. Injectivity is Theorem 19.9.1. Discreteness holds since $k_\sigma \in \mathbb{N}$. □

19.10 Quadratic channel count at each depth

Theorem 19.10.1 (Quadratic channel count (Theorem 14.3.1)). *For a free k-generator triangle-closed system,*

$$\operatorname{rank}(\mathcal{K}_k) = \binom{k}{2}, \qquad \mathcal{K}_k \cong \Lambda^2(\mathbb{Z}^k).$$

Remark 19.10.2 (Dimension versus value). Theorem 19.10.1 gives the rank of $\mathcal{K}_k$. By Theorem 19.5.3, under the inherited second-jet faithfulness condition the scalar channel on admissible defect classes carried by $\mathcal{K}$ agrees, through the realized degree–2 channel attached to the stabilized quadratic carrier, with the scalar curvature projection there. The channel count constrains how many independent defect classes exist at each depth; that conditional scalar-channel identification constrains their values.

19.11 Normalized ratio invariant

The scalar channel Q is unique up to positive normalization (Theorem 15.2.2), so absolute values of $m_0(\sigma)$ depend on a normalization choice. The normalization-independent datum is the ratio.

Definition 19.11.1 (Ratio invariant). Fix a nontrivial reference sector σ_*. The *ratio invariant* is

$$\widehat{m}(\sigma; \sigma_*) := \frac{m_0(\sigma)}{m_0(\sigma_*)} = \frac{Q(\kappa_\sigma)}{Q(\kappa_{\sigma_*})}.$$

Theorem 19.11.2 (Properties of the ratio invariant). *The ratio $\widehat{m}(\sigma; \sigma_*)$ is*

(i) *independent of the normalization of Q;*

(ii) *invariant under admissible transport; and*

(iii) *strictly positive for every nontrivial σ.*

Proof. (i): If $Q' = \lambda Q$ for $\lambda > 0$, numerator and denominator scale by λ, leaving the ratio unchanged. (ii): Both $Q(\kappa_\sigma)$ and $Q(\kappa_{\sigma_*})$ are transport-invariant by Theorem 19.6.3. (iii): The numerator satisfies $m_0(\sigma) > 0$ by Theorem 19.6.2; the denominator satisfies $m_0(\sigma_*) > 0$ by the same theorem. □

Remark 19.11.3 (Relation to the appendix normalized scalar). The ratio $\widehat{m}(\sigma; \sigma_*)$ is the sector-level analogue of the appendix normalized scalar $\widehat{\mathfrak{s}}_{\mathrm{fus}}(u; u_*) = Q(\kappa(u))/Q(\kappa(u_*))$ of section B.10: both cancel the normalization freedom in Q and are preserved by admissible automorphisms.

19.12 Conclusion

The canonical invariant that distinguishes primitive excitation sectors with identical $U(1) \times SU(2) \times SU(3)$ gauge labels is the pair

$$\sigma \longmapsto \big(\epsilon(\kappa_\sigma),\, k_\sigma\big) \in \{+1, -1\} \times \mathbb{N},$$

where $\epsilon(\kappa_\sigma)$ is the orientation parity (Theorem 19.3.3) and k_σ is the refinement depth (Theorem 19.7.1).

Orientation parity reproduces the bosonic/fermionic split of Theorem 19.2.6. Refinement depth separates distinct sectors within a fixed gauge class by Theorems 19.9.1 and 19.9.2.

The scalar invariant $m_0(\sigma) = Q(\kappa_\sigma)$ is strictly positive (Theorem 19.6.2) and transport-invariant (Theorem 19.6.3). Under the inherited second-jet faithfulness condition, its restriction to the integral carrier satisfies $m_0(\sigma) = c\cdot\mathrm{Scal}(\kappa_\sigma)$ by Theorem 19.5.3. The normalization-free version is $\widehat{m}(\sigma; \sigma_*)$ (Theorems 19.11.1 and 19.11.2).

The quadratic channel count $\mathrm{rank}(\mathcal{K}_k) = \binom{k}{2}$ (Theorem 19.10.1) gives the dimension of the space of admissible defect classes at each depth.

Gauge labels determine sector type.
Refinement depth k_σ separates distinct sectors within a fixed gauge class.
Under the inherited second-jet faithfulness condition, $Q\|_{\mathcal{K}} = c \cdot \mathrm{Scal}(\cdot), \quad c > 0.$

Part VII

Numerical Closure

Chapter 20

Conditional Numerical Closure: Quartic Carrier and Mass–Depth Law

20.1 Introduction and logical status

Chapter 19 proves that primitive sectors are classified by internal gauge representation data together with depth information k_σ (Theorems 19.7.1 and 19.9.1), and that the quadratic scalar invariant $m_0(\sigma) = Q(\kappa_\sigma)$ is transport-invariant and strictly positive (Theorems 19.6.2 and 19.6.3). It also records that m_0 is not injective on sectors (Theorem 19.6.4).

This chapter develops a quartic refinement of the quadratic carrier: an invariant depth observable constructed from degree–4 combinatorics. The guiding objective is to separate theorem-level identities from external calibration data, so that each numerical statement is explicitly typed as either a structural consequence or boundary data.

The argument is organized in four components.

(i) A rigidity envelope for admissible scalar depth observables is proved: under intrinsicity and bilinear-origin hypotheses, the only universal leading law is quartic.

(ii) A canonical depth profile $\eta_k \in \Lambda^2(\mathbb{Z}^k)$ is introduced, and its quartic square is computed exactly.

(iii) A quartic depth observable $\mathfrak{m}_4$ is defined and shown to satisfy the exact law $\mathfrak{m}_4(\sigma) = 4\mu\binom{k_\sigma}{4}$.

(iv) External data required for anchor/depth calibration (explicit or recovered), together with cubic selection, are stated explicitly.

Because these numerical identifications are not fixed by the structural closure chain, they are treated as explicit *boundary data* in the sense of chapter A and section A.2. No theorem-level claim is used beyond proved structure plus those stated data.

20.2 Scalar rigidity envelope for depth observables

The attached notes package contributes a useful meta-level rigidity claim: before fixing any explicit formula, admissible scalar depth observables can already be constrained by intrinsicity and transport covariance. This section records that envelope in theorem form and then specializes it to the quartic law used in the remainder of the chapter.

Definition 20.2.1 (Admissible scalar depth observable)**.** A map $\Phi : \mathbb{N} \to \mathbb{R}$ is called an *admissible scalar depth observable* if:

(i) **Intrinsicity:** $\Phi(k)$ depends only on the depth-k quadratic carrier and canonical scalar channel, with no auxiliary coordinate or embedding choice;

(ii) **Transport invariance:** Φ is invariant under admissible transport/gauge identifications on the carrier;

(iii) **Quadratic-bilinear origin:** $\Phi(k)$ is obtained from a scalar contraction of a bilinear expression on the quadratic carrier;

(iv) **Uniform depth covariance:** the assignment $k \mapsto \Phi(k)$ is functorial under canonical depth inclusions and polynomially bounded in k.

Proposition 20.2.2 (Quartic growth envelope)**.** *For every admissible scalar depth observable Φ in the sense of Theorem 20.2.1, there exists a constant $C > 0$ such that*

$$|\Phi(k)| \leq C(1 + k^4) \qquad (k \in \mathbb{N}).$$

Proof. By Theorem 19.10.1, the quadratic carrier rank is $\dim \mathcal{K}_k = \binom{k}{2} = O(k^2)$. A scalar contraction of a bilinear expression on that carrier is therefore controlled by pairings of quadratic directions, hence by $O((\dim \mathcal{K}_k)^2) = O(k^4)$. This is the claimed quartic growth envelope. ☐

Theorem 20.2.3 (Quartic leading-order rigidity under scalar descent)**.** *Assume Φ satisfies Theorem 20.2.1 and additionally:*

(i) $\Phi(0) = 0$*;*

(ii) *odd contributions are eliminated by orientation/phase descent;*

(iii) *the leading asymptotic growth is extensive in* $\binom{k}{2}^2$.

Then there exists $\alpha > 0$ such that

$$\Phi(k) = \alpha k^4 + o(k^4) \qquad (k \to \infty).$$

In particular, the universal leading law is quartic.

Proof. By Theorem 20.2.2, $\Phi(k) = O(k^4)$. The hypothesized extensivity in $\binom{k}{2}^2$ identifies the leading asymptotic behavior with a positive multiple of $\binom{k}{2}^2 \sim \frac{1}{4}k^4$, so there exists $\alpha > 0$ such that $\Phi(k) = \alpha k^4 + o(k^4)$. The orientation/phase descent hypothesis ensures that no odd-order leading contribution survives, but no stronger conclusion about lower even-order terms is claimed here. ☐

Remark 20.2.4 (Role of the cubic selector). The rigidity theorem isolates the universal leading behavior of the depth law. It does *not* derive a selector for admissible depths. As in the notes package, cubic conditions act as domain restrictions on admissible sectors, while quartic rigidity fixes the depth scaling once a sector is admissible.

20.3 Depth profiles in the quadratic carrier

By Theorems 14.3.1 and 19.10.1, the stage-k quadratic carrier is identified with $\Lambda^2(\mathbb{Z}^k)$. Fix the ordered basis $e_1, \ldots, e_k$ of $\mathbb{Z}^k$.

Definition 20.3.1 (Cumulative depth profile)**.** For $k \in \mathbb{N}$, define

$$\eta_k := \sum_{1 \le i < j \le k} e_i \wedge e_j \in \Lambda^2(\mathbb{Z}^k).$$

Remark 20.3.2 (Quadratic count compatibility). The number of summands in η_k is $\binom{k}{2}$, matching the quadratic channel count of Theorem 14.3.1 and the normalization variable used in Theorem 14.4.1.

Definition 20.3.3 (Quartic profile)**.** Define

$$\omega_k := \eta_k \wedge \eta_k \in \Lambda^4(\mathbb{Z}^k).$$

Lemma 20.3.4 (Explicit quartic expansion)**.** *For every $k \ge 4$,*

$$\omega_k = 2 \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le k} e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}.$$

Proof. Write $u_{ab} := e_a \wedge e_b$ for $a < b$. Then

$$\omega_k = \sum_{a<b} \sum_{c<d} u_{ab} \wedge u_{cd}.$$

If $\{a, b\} \cap \{c, d\} \neq \varnothing$, the wedge term vanishes. Hence only disjoint pairs contribute.

Fix $i_1 < i_2 < i_3 < i_4$. The nonzero contributions using exactly these four indices come from the three unordered pairings:

$$(i_1 i_2)(i_3 i_4), \qquad (i_1 i_3)(i_2 i_4), \qquad (i_1 i_4)(i_2 i_3).$$

Relative to the oriented basis element $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$, their signs are $+1, -1, +1$, so the signed sum is 1. Because degree–2 forms commute under wedge ($\alpha \wedge \beta = \beta \wedge \alpha$ for $\deg \alpha = \deg \beta = 2$), each unordered pairing appears twice in the ordered double sum. Therefore the total coefficient of each ordered 4-tuple is $2 \cdot 1 = 2$, proving the formula. ☐

Definition 20.3.5 (Quartic coefficient energy)**.** For $\omega = \sum_I c_I e_I \in \Lambda^4(\mathbb{R}^k)$, where $I = (i_1 < i_2 < i_3 < i_4)$ and $e_I := e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$, define

$$E_4(\omega) := \sum_I c_I^2.$$

Proposition 20.3.6 (Quartic counting law)**.** *For $k \geq 4$,*

$$E_4(\omega_k) = 4\binom{k}{4}.$$

Proof. By Theorem 20.3.4, every basis 4-blade appears with coefficient 2, and there are exactly $\binom{k}{4}$ such blades. Hence

$$E_4(\omega_k) = \binom{k}{4} \cdot 2^2 = 4\binom{k}{4}.$$

□

20.4 Quartic depth observable and mass-coordinate law

Definition 20.4.1 (Quartic depth observable)**.** Fix $\mu > 0$. For a nontrivial primitive sector σ with depth k_σ (Theorem 19.7.1), define

$$\mathfrak{m}_4(\sigma) := \mu\, E_4\big(\omega_{k_\sigma}\big) = \mu\, E_4\big(\eta_{k_\sigma} \wedge \eta_{k_\sigma}\big).$$

Theorem 20.4.2 (Exact quartic formula)**.** *For every nontrivial primitive sector σ,*

$$\mathfrak{m}_4(\sigma) = 4\mu\binom{k_\sigma}{4}.$$

Proof. Combine Theorem 20.4.1 with Theorem 20.3.6 evaluated at $k = k_\sigma$. □

Theorem 20.4.3 (Intrinsic quartic mass-coordinate bridge)**.** *For every nontrivial primitive sector σ, the closed-system mass coordinate determined by the quartic coefficient-energy readout is*

$$m_\sigma := \mathfrak{m}_4(\sigma).$$

Proof. The closed-system scalar readout available at quartic depth is the coefficient energy of the canonical quartic profile ω_{k_σ}, normalized by the universal positive scale μ. By Theorem 20.4.1, that readout is exactly

$$\mu E_4(\omega_{k_\sigma}) = \mathfrak{m}_4(\sigma).$$

Thus the mass coordinate internal to the closed-system numerical package is not an extra datum; it is the quartic coefficient-energy readout already constructed from the stabilized carrier. □

Corollary 20.4.4 (Mass-coordinate law)**.** *For every nontrivial primitive sector,*

$$m_\sigma = 4\mu\binom{k_\sigma}{4}.$$

Proof. Immediate from Theorem 20.4.2. ☐

Remark 20.4.5 (What is forced vs. what is calibrated). The combinatorial identity $\mathfrak{m}_4(\sigma) = 4\mu\binom{k_\sigma}{4}$ is theorem-level once $\mathfrak{m}_4$ is defined. The closed-system mass coordinate is therefore theorem-level. Identifying that internal coordinate with an external measured mass convention is an observer-side semantic step, not an additional internal axiom.

20.5 Depth anchors and ratio package

Definition 20.5.1 (Explicit depth-anchor package)**.** The explicit $6, 18, 35$ depth-anchor package is the declaration that the three reference sectors e, μ, τ have depths

$$k_e = 6, \qquad k_\mu = 18, \qquad k_\tau = 35.$$

Corollary 20.5.2 (Conditional ratio predictions)**.** *For the closed-system mass coordinate and the explicit anchor package of Theorem 20.5.1,*

$$\frac{m_\mu}{m_e} = \frac{\binom{18}{4}}{\binom{6}{4}} = 204,$$

$$\frac{m_\tau}{m_e} = \frac{\binom{35}{4}}{\binom{6}{4}} = \frac{10472}{3} \approx 3490.67.$$

Proof. In mass ratios, the normalization constant 4μ cancels. Substitute the depth values from Theorem 20.5.1 into Theorem 20.4.4. ☐

20.6 Cubic obstruction from phase typing (optional selector data)

This section introduces an internal phase-typing selector for admissible depth profiles. This mechanism is not derived in Chapters 1–19, so it is recorded here as an explicit optional data package.

Definition 20.6.1 (Phase data at depth k)**.** Fix a charge unit $q_0 \in \mathbb{N}_{>0}$. For a depth-k profile choose integer labels $q_1, \ldots, q_k \in \mathbb{Z}$, and define phase factors

$$\phi_i := \exp\left(\frac{2\pi i}{q_0}\, q_i\right) \qquad (1 \le i \le k).$$

Definition 20.6.2 (Phase admissibility)**.** A depth-k phase configuration is *admissible* if it extends to a coherent phase assignment on all higher refinement levels.

Lemma 20.6.3 (Triple coherence)**.** *In this optional phase-typing model, admissibility requires triviality of the* 3*-cocycle*

$$\Phi(i,j,\ell) := \exp\left(\frac{2\pi i}{q_0^3}\, q_i q_j q_\ell\right).$$

Consequently,

$$\sum_{1\le i<j<\ell\le k} q_i q_j q_\ell = 0.$$

Proof. Refinement invariance forces associativity of phase composition on triple overlaps. This produces a 3-cocycle obstruction represented by $\Phi(i,j,\ell)$, and admissibility requires that obstruction class to be trivial. In the present phase package, this is encoded by vanishing of the cubic triple sum. □

Definition 20.6.4 (Cubic charge defect)**.** Given integer labels $q_1, \dots, q_k$, define

$$\Delta_3(k) := \sum_{i=1}^{k} q_i^3.$$

Lemma 20.6.5 (Cubic obstruction under charge neutrality)**.** *Assume*

$$\sum_{i=1}^{k} q_i = 0.$$

If

$$\sum_{1\le i<j<\ell\le k} q_i q_j q_\ell = 0,$$

then

$$\Delta_3(k) = 0.$$

Proof. Let $p_r := \sum_i q_i^r$ and e_r be the elementary symmetric sums. Newton's identity gives

$$p_3 = e_1^3 - 3e_1e_2 + 3e_3.$$

Under charge neutrality, $e_1 = p_1 = \sum_i q_i = 0$, so $p_3 = 3e_3$. Therefore $e_3 = \sum_{i<j<\ell} q_i q_j q_\ell = 0$ implies $p_3 = \sum_i q_i^3 = 0$, which is exactly $\Delta_3(k) = 0$. □

Corollary 20.6.6 (Cubic selector induced by phase typing)**.** *Under phase admissibility and charge neutrality,*

$$\Delta_3(k) = 0$$

is a necessary condition for depth-k *admissibility.*

Proof. Combine Theorems 20.6.3 and 20.6.5. □

Proposition 20.6.7 (Minimal depth from cubic cancellation)**.** *Assume Theorem 20.6.6 and suppose*

$$\Delta_3(34) \neq 0, \qquad \Delta_3(35) = 0.$$

Then the minimal admissible depth satisfying the cubic criterion is $k = 35$.

Proof. By Theorem 20.6.6, admissibility at depth k requires $\Delta_3(k) = 0$. The given values rule out $k = 34$ and admit $k = 35$. Hence the least admissible depth under this criterion is 35. □

Corollary 20.6.8 (Conditional third-generation depth)**.** *If the third-generation onset depth is identified with the first depth admissible under the cubic phase-typing criterion, then $k_\tau = 35$.*

20.7 Structural dependency map with the closure chain

For traceability against the preceding chapters, this section records the exact structural interfaces used by the numerical analysis.

Definition 20.7.1 (Dependency interface)**.** The Chapter 20 dependency interface consists of the following four dependencies:

(i) **Depth typing and invariance.** Primitive sectors carry a depth index k_σ that is transport-invariant (Theorems 19.7.1 and 19.7.3).

(ii) **Depth separation and hierarchy.** Distinct depth values represent distinct hierarchy levels (Theorems 19.9.1 and 19.9.2).

(iii) **Quadratic carrier structure.** The stabilized degree–2 carrier is identified with $\Lambda^2(\mathbb{Z}^k)$, with rank $\binom{k}{2}$ (Theorems 14.3.1 and 19.10.1).

(iv) **Normalization ledger.** Quantitatively open entries are tracked as boundary data (chapter A and section A.2).

Proposition 20.7.2 (Logical role of the dependency interface)**.** *The unconditional quartic-combinatorics package in this chapter is a formal consequence of Theorem 20.7.1 and multilinear algebra on $\Lambda^2(\mathbb{Z}^k)$. The scalar rigidity-envelope results and the optional cubic-selector results use additional chapter-local hypotheses, recorded explicitly in Theorems 20.2.1 and 20.6.2. Concrete generation-depth calibration is conditional on explicit calibration data: either fixed anchor depths Theorem 20.5.1, or the ratio/cubic recovery route of Theorem 20.14.5.*

Proof. Depth labels k_σ and their invariance enter through Theorems 19.7.1 and 19.7.3, and the carrier used for quartic combinatorics is fixed by Theorems 14.3.1 and 19.10.1. Therefore the identities involving η_k, ω_k, coefficient counts, the polynomial $q(k) = 4\binom{k}{4}$, and their derived ratio, monotonicity, inversion, and scaling consequences are structural. The scalar rigidity-envelope package additionally uses the local admissibility hypotheses of Theorem 20.2.1, and the optional cubic selector package uses the separately stated phase-admissibility and charge-neutrality hypotheses recorded in Theorems 20.6.2, 20.6.5 and 20.16.3. The transition from these structural or locally hypothesized identities to closed-system mass coordinates uses the theorem-level bridge $m_\sigma = \mathfrak{m}_4(\sigma)$ (Theorem 20.4.3), and concrete anchor calibration uses either explicit depth anchors (Theorem 20.5.1) or the ratio/cubic recovery route (Theorem 20.14.5). The observer-side interpretation of these closed-system coordinates is tracked separately in section A.2. Hence the claimed logical split follows. □

Definition 20.7.3 (Chapter 20 numerical output types)**.** A Chapter 20 numerical output is any statement of one of the following forms:

(O1) an exact quartic coefficient-count or polynomial identity for $E_4(\omega_k)$, $q(k) = 4\binom{k}{4}$, or their finite differences;

(O2) a closed-system mass-coordinate identity obtained from $m_\sigma = \mathfrak{m}_4(\sigma)$;

(O3) an order, convexity, scaling, or finite-inversion consequence of $q(k)$;

(O4) an anchor-ratio consequence for a specified depth-anchor package or for the ratio/cubic recovery route;

(O5) a cubic-selector consequence using the separately specified phase-typing data.

Theorem 20.7.4 (Numerical dependency normal form)**.** *Every Chapter 20 numerical output factors through the ordered dependency tuple*

$$\mathcal{D}_{20} := \big(k_\sigma, \Lambda^2(\mathbb{Z}^{k_\sigma}), \eta_{k_\sigma}, \mu, \mathcal{A}, \Delta_3\big),$$

where $\mathcal{A}$ denotes either an explicitly specified anchor package or the ratio/cubic recovery data, and Δ_3 denotes optional cubic-selector data. More precisely:

(i) *outputs of type* (O1) *use only* $(k_\sigma, \Lambda^2(\mathbb{Z}^{k_\sigma}), \eta_{k_\sigma})$*;*

(ii) *outputs of types* (O2) *and* (O3) *use those structural data together with the positive scale* μ*;*

(iii) *outputs of type* (O4) *additionally use exactly the anchor data named in the relevant statement;*

(iv) *outputs of type* (O5) *additionally use exactly the phase-typing data named in the relevant statement.*

No further numerical datum is used implicitly.

Proof. For (O1), the coefficient computation is Theorems 20.3.6 and 20.8.6; its proof uses only the canonical exterior-square carrier and the profile η_k. The difference tower, recurrence, monotonicity, convexity, and elementary binomial estimates are then algebraic consequences of the same polynomial $q(k) = 4\binom{k}{4}$.

For (O2), Theorem 20.4.3 identifies the closed-system mass coordinate with $\mathfrak{m}_4$, and Theorem 20.4.4 substitutes the exact quartic formula; the only additional scalar datum is the positive normalization μ already present in Theorem 20.4.1. Type (O3) statements are obtained from the same formula by applying the order and inversion properties of $q(k)$, so no new datum enters.

For (O4), Theorem 20.14.2 cancels the common factor 4μ and leaves only the specified anchor depths. When those depths are not stipulated, Theorem 20.14.5 supplies them from the explicitly named ratio/cubic recovery data. For (O5), the cubic statements cite Theorems 20.6.2 and 20.6.6 and therefore use exactly the phase-typing data displayed there.

These cases exhaust Theorem 20.7.3. Each proof in the chapter cites one of the structural, mass-coordinate, anchor, or cubic routes listed above; hence any dependency not appearing in $\mathcal{D}_{20}$ would have no cited entry point. Therefore no further numerical datum is used implicitly. □

Corollary 20.7.5 (Auditability of Chapter 20 outputs)**.** *To audit any Chapter 20 numerical claim, it is enough to identify its output type in Theorem 20.7.3 and then read off the corresponding component of $\mathcal{D}_{20}$ from Theorem 20.7.4.*

Proof. The theorem gives an exhaustive case split by output type and records the exact dependency components used in each case. Thus checking the output type determines the complete dependency list. □

Remark 20.7.6 (Consistency with Chapter 19 scope). Chapter 19 establishes positivity and transport invariance for m_0, together with non-injectivity of m_0 (Theorems 19.6.2 to 19.6.4). The quartic observable $\mathfrak{m}_4$ introduced here is not asserted to replace these structural facts; it is an additional depth-based statistic with separate conditional interpretation.

20.8 Coefficient-level decomposition of the quartic profile

This section expands Theorem 20.3.4 into a fully indexed coefficient computation.

Definition 20.8.1 (Support of a wedge monomial)**.** For $a < b$, define $u_{ab} := e_a \wedge e_b$. For (a, b, c, d) with $a < b$, $c < d$, define the support set

$$\operatorname{supp}(a, b; c, d) := \{a, b, c, d\}.$$

Lemma 20.8.2 (Support criterion for nonzero terms). *For $a < b$, $c < d$, the wedge term $u_{ab} \wedge u_{cd}$ is nonzero if and only if $|\operatorname{supp}(a, b; c, d)| = 4$.*

Proof. If $\operatorname{supp}(a, b; c, d)$ has cardinality strictly smaller than 4, then one index appears at least twice in $e_a \wedge e_b \wedge e_c \wedge e_d$, hence alternatingness of exterior product implies $u_{ab} \wedge u_{cd} = 0$.

Conversely, if $|\operatorname{supp}(a, b; c, d)| = 4$, then e_a, e_b, e_c, e_d are distinct basis vectors, so their wedge is a nonzero multiple of the oriented basis element determined by the sorted index order. Thus $u_{ab} \wedge u_{cd} \neq 0$. □

Definition 20.8.3 (Coefficient extraction). For each ordered 4-subset $I = (i_1 < i_2 < i_3 < i_4)$, define $e_I := e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$. If $\omega = \sum_I c_I e_I$, denote by $\operatorname{coeff}_I(\omega) := c_I$ the I-coefficient.

Lemma 20.8.4 (Pairing-sign table on a fixed support). *Fix $I = (i_1 < i_2 < i_3 < i_4)$. Relative to e_I, the three unordered pairings satisfy*

$$u_{i_1 i_2} \wedge u_{i_3 i_4} = +e_I,$$

$$u_{i_1 i_3} \wedge u_{i_2 i_4} = -e_I,$$

$$u_{i_1 i_4} \wedge u_{i_2 i_3} = +e_I.$$

Hence their signed sum equals e_I.

Proof. The first identity is immediate from ordering. For the second,

$$e_{i_1} \wedge e_{i_3} \wedge e_{i_2} \wedge e_{i_4} = -e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4} = -e_I,$$

since one transposition exchanges i_3 and i_2. For the third,

$$e_{i_1} \wedge e_{i_4} \wedge e_{i_2} \wedge e_{i_3} = (+1)e_I,$$

because the permutation $(1, 4, 2, 3) \to (1, 2, 3, 4)$ has two inversions. Summing gives the claim. □

Proposition 20.8.5 (Uniform coefficient law). *For every ordered 4-subset I, one has*

$$\operatorname{coeff}_I(\omega_k) = 2.$$

Equivalently,

$$\omega_k = 2 \sum_I e_I,$$

where I runs through all ordered 4-subsets of $\{1, \dots, k\}$.

Proof. Expand $\omega_k = \sum_{a<b} \sum_{c<d} u_{ab} \wedge u_{cd}$. By Theorem 20.8.2, only terms with four distinct indices contribute. Fix I. Within support I, the unordered contribution equals e_I by Theorem 20.8.4. Because degree–2 forms commute, $u_{ab} \wedge u_{cd} = u_{cd} \wedge u_{ab}$, each unordered pairing appears twice in the ordered double sum. Hence $\operatorname{coeff}_I(\omega_k) = 2$. Summing over I yields the displayed expansion. □

Corollary 20.8.6 (Energy identity by coefficient counting). *For $k \geq 4$,*

$$E_4(\omega_k) = \sum_I \operatorname{coeff}_I(\omega_k)^2 = \sum_I 4 = 4\binom{k}{4}.$$

Proof. Apply Theorem 20.8.5 and count the number of ordered 4-subsets. □

20.9 Polynomial form and discrete quartic calculus

Definition 20.9.1 (Reduced quartic depth polynomial). Define

$$q(k) := 4\binom{k}{4},$$

so that $\mathfrak{m}_4(\sigma) = \mu\, q(k_\sigma)$ by Theorem 20.4.2.

Proposition 20.9.2 (Closed polynomial form). *For all integers k,*

$$q(k) = \frac{1}{6}k(k-1)(k-2)(k-3).$$

Proof. By definition,

$$q(k) = 4\binom{k}{4} = 4\frac{k(k-1)(k-2)(k-3)}{24},$$

which simplifies to the displayed expression. □

Lemma 20.9.3 (Vanishing order at low depth). *One has*

$$q(0) = q(1) = q(2) = q(3) = 0, \qquad q(4) = 4.$$

Proof. The factorized form from Theorem 20.9.2 contains the four consecutive factors k, $k-1$, $k-2$, $k-3$, so the first four values vanish. At $k = 4$, $q(4) = 4\binom{4}{4} = 4$. □

Definition 20.9.4 (Forward difference operator). For a function $f : \mathbb{Z} \to \mathbb{R}$, define

$$(\Delta f)(k) := f(k+1) - f(k).$$

Inductively, $\Delta^{r+1} f := \Delta(\Delta^r f)$.

Theorem 20.9.5 (Finite-difference tower for q). *For every integer k, one has*

$$\Delta q(k) = 4\binom{k}{3},$$

$$\Delta^2 q(k) = 4\binom{k}{2},$$

$$\Delta^3 q(k) = 4k,$$

$$\Delta^4 q(k) = 4,$$

$$\Delta^5 q(k) = 0.$$

Proof. Use Pascal's identity $\binom{k+1}{r} - \binom{k}{r} = \binom{k}{r-1}$. Then

$$\Delta q(k) = 4\left(\binom{k+1}{4} - \binom{k}{4}\right) = 4\binom{k}{3}.$$

Applying Δ repeatedly gives

$$\Delta^2 q(k) = 4\binom{k}{2},$$

$$\Delta^3 q(k) = 4\binom{k}{1} = 4k,$$

$$\Delta^4 q(k) = 4\binom{k}{0} = 4,$$

$$\Delta^5 q(k) = 0.$$

□

Corollary 20.9.6 (Order–5 linear recurrence)**.** *The sequence $q(k)$ satisfies*

$$q(k+5) - 5q(k+4) + 10q(k+3) - 10q(k+2) + 5q(k+1) - q(k) = 0.$$

Proof. The left-hand side is exactly $\Delta^5 q(k)$, which vanishes by Theorem 20.9.5. □

Remark 20.9.7 (Rigidity viewpoint). The identity $\Delta^4 q \equiv 4$ characterizes q among sequences with zero initial values at depths $0, 1, 2, 3$: the quartic law is the unique depth-polynomial with constant fourth difference equal to 4.

20.10 Monotonicity, convexity, and order transfer

Proposition 20.10.1 (Strict growth)**.** *For all $k \geq 4$,*

$$q(k+1) - q(k) = 4\binom{k}{3} > 0.$$

Hence q is strictly increasing on $\{4, 5, 6, \dots\}$.

Proof. The difference formula is Theorem 20.9.5. For $k \geq 4$, $\binom{k}{3} > 0$, so the increment is positive. □

Proposition 20.10.2 (Discrete convexity)**.** *For all $k \geq 2$,*

$$\Delta^2 q(k) = 4\binom{k}{2} > 0.$$

Thus the increments $q(k+1) - q(k)$ are strictly increasing.

Proof. By Theorem 20.9.5, $\Delta^2 q(k) = 4\binom{k}{2}$. For $k \geq 2$, $\binom{k}{2} > 0$. Therefore Δq is strictly increasing. □

Theorem 20.10.3 (Depth order transfers to mass order)**.** *For the closed-system mass coordinate of Theorem 20.4.3, let σ, τ be nontrivial primitive sectors with $k_\sigma, k_\tau \geq 4$. If*

$$k_\sigma < k_\tau,$$

then

$$m_\sigma < m_\tau.$$

Proof. By Theorem 20.4.4, $m_x = \mu q(k_x)$ for $x \in \{\sigma, \tau\}$, with $\mu > 0$. By Theorem 20.10.1, $q(k_\sigma) < q(k_\tau)$. Multiplication by positive μ preserves strict inequality. □

Corollary 20.10.4 (Injectivity on realized depth values)**.** *For the closed-system mass coordinate, the map $k \mapsto m(k) := 4\mu\binom{k}{4}$ is injective on $\{4, 5, 6, \dots\}$.*

Proof. If $m(k_1) = m(k_2)$, strict monotonicity from Theorem 20.10.1 implies $k_1 = k_2$. □

20.11 Inversion bounds for the mass-coordinate law

Proposition 20.11.1 (Elementary quartic bounds for $\binom{k}{4}$)**.** *For every integer $k \geq 4$,*

$$(k-3)^4 \leq k(k-1)(k-2)(k-3) \leq k^4.$$

Equivalently,

$$\frac{(k-3)^4}{24} \leq \binom{k}{4} \leq \frac{k^4}{24}.$$

Proof. For $k \geq 4$, each factor in $k(k-1)(k-2)(k-3)$ lies between $k-3$ and k, so the product lies between the corresponding fourth powers. Division by 24 gives the binomial form. □

Theorem 20.11.2 (Depth window from one mass value)**.** *For the closed-system mass coordinate of Theorem 20.4.3, let $m > 0$ satisfy*

$$m = 4\mu\binom{k}{4}$$

for some integer $k \geq 4$, and define

$$x := \left(\frac{6m}{\mu}\right)^{1/4}.$$

Then

$$k - 3 \leq x \leq k,$$

hence

$$x \leq k \leq x + 3.$$

Proof. From the mass law,

$$\frac{6m}{\mu} = k(k-1)(k-2)(k-3).$$

Apply Theorem 20.11.1:

$$(k-3)^4 \le \frac{6m}{\mu} \le k^4.$$

Taking fourth roots gives $k-3 \le x \le k$, and rearrangement gives $x \le k \le x+3$. □

Corollary 20.11.3 (Finite-candidate inversion algorithm)**.** *For the closed-system mass coordinate, a mass-coordinate value determines depth by checking at most four consecutive integers.*

Proof. By Theorem 20.11.2, any admissible integer depth must lie in $[x, x+3]$, which contains at most four integers. By Theorem 20.10.1, at most one of these candidates can match the observed mass. □

Proposition 20.11.4 (Asymptotic depth extraction)**.** *For the closed-system mass coordinate, for large depth values one has*

$$k = \left(\frac{6m}{\mu}\right)^{1/4} + O(1).$$

Proof. This is immediate from Theorem 20.11.2. The difference between k and the quartic root is always bounded by 3. □

20.12 Coupling to the forced quadratic normalization

Definition 20.12.1 (Quadratic and quartic carriers at depth k)**.** Define

$$M_2(k) := \binom{k}{2},$$

$$M_4(k) := 4\mu\binom{k}{4}.$$

Here M_2 is the structural quadratic count (Theorem 14.3.1), and M_4 is the quartic mass-coordinate law.

Proposition 20.12.2 (Exact ratio formula M_4/M_2^2)**.** *For every* $k \ge 4$,

$$\frac{M_4(k)}{M_2(k)^2} = \frac{2\mu}{3} \cdot \frac{(k-2)(k-3)}{k(k-1)}.$$

Proof. Using Theorem 20.12.1,

$$\frac{M_4(k)}{M_2(k)^2} = \frac{4\mu\binom{k}{4}}{\binom{k}{2}^2} = 4\mu\frac{\frac{k(k-1)(k-2)(k-3)}{24}}{\frac{k^2(k-1)^2}{4}}.$$

Simplifying yields

$$\frac{2\mu}{3}\frac{(k-2)(k-3)}{k(k-1)}.$$

□

Corollary 20.12.3 (Asymptotic quadratic coupling). *As $k \to \infty$,*

$$M_4(k) \sim \frac{2\mu}{3}\, M_2(k)^2.$$

Proof. In Theorem 20.12.2, the rational factor $\frac{(k-2)(k-3)}{k(k-1)}$ tends to 1. Therefore $M_4/M_2^2 \to 2\mu/3$. □

Remark 20.12.4 (Interpretive consequence). Under the same depth variable, the quartic law is asymptotically quadratic in the quadratic carrier count. Thus the Chapter 20 observable is not an unrelated scalar; it is a higher-order combinatorial functional of the Chapter 14 carrier size.

20.13 Filtration-time scaling under forced normalization

Definition 20.13.1 (Time-lifted quartic observable). Let $k(T)$ denote activated depth at filtration time T, as in Theorem 14.4.1. Define

$$M_4(T) := 4\mu\binom{k(T)}{4}.$$

Theorem 20.13.2 (Quadratic-time scaling of the quartic observable). *Under the canonical normalization of Theorems 14.4.1 and 14.4.2, one has*

$$M_4(T) \asymp T^2.$$

Proof. By Theorem 14.4.2, $k(T) \asymp \sqrt{T}$. By Theorem 20.9.2, $\binom{k}{4} \asymp k^4$ for large k. Therefore

$$M_4(T) = 4\mu\binom{k(T)}{4} \asymp k(T)^4 \asymp (\sqrt{T})^4 = T^2.$$

□

Corollary 20.13.3 (Separation of visible scales). *With $M_2(T) \asymp T$ from Theorem 14.4.1, the quartic observable satisfies*

$$M_4(T) \asymp M_2(T)^2.$$

Proof. Combine Theorem 14.4.1 with Theorem 20.13.2. □

20.14 General anchor formalism and ratio geometry

Definition 20.14.1 (Anchor package)**.** An anchor package is a tuple

$$\mathcal{A} = (k_1, \ldots, k_r),$$

with integers $k_i \geq 4$, together with the mass-coordinate bridge theorem Theorem 20.4.3.

Theorem 20.14.2 (General ratio law)**.** *Under Theorem 20.14.1, for any indices i, j,*

$$\frac{m_i}{m_j} = \frac{\binom{k_i}{4}}{\binom{k_j}{4}}.$$

Proof. By Theorem 20.4.4, $m_i = 4\mu\binom{k_i}{4}$ and similarly for m_j. Divide and cancel the common factor 4μ. □

Proposition 20.14.3 (Multiplicative transitivity of ratio data)**.** *For any i, j, ℓ,*

$$\frac{m_i}{m_j} \cdot \frac{m_j}{m_\ell} = \frac{m_i}{m_\ell}.$$

On the depth side this reads

$$\frac{\binom{k_i}{4}}{\binom{k_j}{4}} \cdot \frac{\binom{k_j}{4}}{\binom{k_\ell}{4}} = \frac{\binom{k_i}{4}}{\binom{k_\ell}{4}}.$$

Proof. This is immediate in the multiplicative group $\mathbb{R}_{>0}$, or directly from Theorem 20.14.2. □

Corollary 20.14.4 (Recovery of the $(6, 18, 35)$ package)**.** *For the explicit depth-anchor package of Theorem 20.5.1,*

$$\frac{m_\mu}{m_e} = \frac{\binom{18}{4}}{\binom{6}{4}} = 204,$$

$$\frac{m_\tau}{m_e} = \frac{\binom{35}{4}}{\binom{6}{4}} = \frac{10472}{3}.$$

Proof. Apply Theorem 20.14.2 to $(k_e, k_\mu, k_\tau) = (6, 18, 35)$, then evaluate binomial coefficients. □

The explicit $(6, 18, 35)$ anchor package can also be recovered from ratio data once third-generation onset depth is fixed by the cubic selector route. Dependency basis: Theorems 20.6.8, 20.10.1 and 20.14.2.

Theorem 20.14.5 (Ratio/cubic closure of the $(6, 18, 35)$ anchors)**.** *For the closed-system mass coordinate of Theorem 20.4.3, let e, μ, τ be reference sectors with depths $k_e, k_\mu, k_\tau \geq 4$. Suppose the following calibration data hold:*

(i) $k_\tau = 35$ *(for instance by Theorem 20.6.8);*

(ii) *measured mass ratios satisfy*

$$\frac{m_\mu}{m_e} = 204, \qquad \frac{m_\tau}{m_e} = \frac{10472}{3}.$$

Then

$$k_e = 6, \qquad k_\mu = 18, \qquad k_\tau = 35.$$

In particular, the explicit depth-anchor package of Theorem 20.5.1 is recovered.

Proof. From Theorem 20.14.2 and $k_\tau = 35$,

$$\frac{\binom{35}{4}}{\binom{k_e}{4}} = \frac{m_\tau}{m_e} = \frac{10472}{3}.$$

Hence

$$\binom{k_e}{4} = \frac{\binom{35}{4}}{10472/3} = 15 = \binom{6}{4}.$$

By Theorem 20.10.1, $k \mapsto q(k) = 4\binom{k}{4}$ is strictly increasing on $\{4, 5, 6, \dots\}$, so $k \mapsto \binom{k}{4}$ is strictly increasing there; therefore $k_e = 6$.

Applying Theorem 20.14.2 to (μ, e),

$$\frac{\binom{k_\mu}{4}}{\binom{6}{4}} = \frac{m_\mu}{m_e} = 204,$$

so

$$\binom{k_\mu}{4} = 204 \cdot \binom{6}{4} = 204 \cdot 15 = 3060 = \binom{18}{4}.$$

Strict monotonicity again yields $k_\mu = 18$, and $k_\tau = 35$ is the given cubic-onset depth. Thus $(k_e, k_\mu, k_\tau) = (6, 18, 35)$, i.e. the explicit depth-anchor package of Theorem 20.5.1 is recovered. □

Corollary 20.14.6 (Discharge of explicit depth-anchor package)**.** *Whenever Theorem 20.14.5 applies, consequences attached to the explicit depth-anchor package of Theorem 20.5.1 can be used without stipulating that package independently.*

Proof. Immediate from Theorem 20.14.5. □

20.15 Local ratio sensitivity and depth identifiability

Proposition 20.15.1 (One-step ratio formula)**.** *For integer $k \geq 4$,*

$$\frac{m(k+1)}{m(k)} = \frac{\binom{k+1}{4}}{\binom{k}{4}} = \frac{k+1}{k-3},$$

where $m(k) := 4\mu\binom{k}{4}$.

Proof. Compute directly:

$$\frac{\binom{k+1}{4}}{\binom{k}{4}} = \frac{(k+1)k(k-1)(k-2)}{k(k-1)(k-2)(k-3)} = \frac{k+1}{k-3}.$$

□

Corollary 20.15.2 (Closed-form depth from successive ratio)**.** *If a measured consecutive ratio is*

$$r := \frac{m(k+1)}{m(k)} > 1,$$

then

$$k = \frac{3r+1}{r-1}.$$

Proof. By Theorem 20.15.1, $r(k-3) = k+1$. Rearrange: $k(r-1) = 3r+1$, hence the formula. □

Proposition 20.15.3 (Relative sensitivity to a ± 1 depth shift)**.** *For $k \geq 5$,*

$$\frac{m(k+1) - m(k)}{m(k)} = \frac{4}{k-3},$$

$$\frac{m(k) - m(k-1)}{m(k)} = \frac{4}{k}.$$

Thus the relative one-step sensitivity decays like $O(1/k)$.

Proof. From Theorem 20.15.1,

$$\frac{m(k+1) - m(k)}{m(k)} = \frac{m(k+1)}{m(k)} - 1 = \frac{k+1}{k-3} - 1 = \frac{4}{k-3}.$$

Similarly,

$$\frac{m(k-1)}{m(k)} = \frac{\binom{k-1}{4}}{\binom{k}{4}} = \frac{k-4}{k},$$

so

$$\frac{m(k) - m(k-1)}{m(k)} = 1 - \frac{k-4}{k} = \frac{4}{k}.$$

□

Remark 20.15.4 (Practical calibration implication). At larger anchor depths, fixed absolute uncertainty in depth produces smaller relative distortion in mass ratios. This is a direct consequence of Theorem 20.15.3.

20.16 Cubic selector in symmetric-polynomial normal form

Definition 20.16.1 (Power sums and elementary sums)**.** For charges $q_1, \ldots, q_k$, define

$$p_r := \sum_{i=1}^{k} q_i^r \qquad (r \geq 1),$$

and denote by e_r the elementary symmetric polynomials. In this notation,

$$p_1 = e_1, \qquad e_3 = \sum_{1 \leq i < j < \ell \leq k} q_i q_j q_\ell.$$

Proposition 20.16.2 (Neutrality identity)**.** *If $p_1 = 0$, then*

$$p_3 = 3e_3.$$

In particular,

$$\sum_{i=1}^{k} q_i^3 = 3 \sum_{1 \leq i < j < \ell \leq k} q_i q_j q_\ell.$$

Proof. Newton's identity gives

$$p_3 = e_1 p_2 - e_2 p_1 + 3e_3.$$

Since $p_1 = e_1 = 0$, the first two terms vanish, yielding $p_3 = 3e_3$. □

Theorem 20.16.3 (Equivalent cubic selector forms)**.** *Assume charge neutrality $\sum_i q_i = 0$. Then the following are equivalent:*

(i) $\sum_{i<j<\ell} q_i q_j q_\ell = 0$*;*

(ii) $\sum_i q_i^3 = 0$*;*

(iii) $\Delta_3(k) = 0$*.*

Proof. By Theorem 20.16.2, $\sum_i q_i^3 = 3 \sum_{i<j<\ell} q_i q_j q_\ell$. Thus (i) and (ii) are equivalent. Condition (iii) is exactly the notation of (ii) by Theorem 20.6.4. □

Proposition 20.16.4 (General minimal-depth criterion)**.** *Suppose admissibility requires $\Delta_3(k) = 0$, and there exists $k_* \geq 4$ such that*

$$\Delta_3(k) \neq 0 \quad \textit{for all } 4 \leq k < k_*,$$

$$\Delta_3(k_*) = 0.$$

Then the minimal admissible depth is k_*.

Proof. The first line excludes every depth smaller than k_*. The second line certifies admissibility at k_*. Therefore no admissible depth exists below k_*, and k_* is the least admissible depth. □

Corollary 20.16.5 (Specialization to $k_\tau = 35$)**.** *If*

$$\Delta_3(k) \neq 0 \text{ for all } 4 \leq k \leq 34,$$

$$\Delta_3(35) = 0,$$

then the first cubic-admissible depth is 35*.*

Proof. Apply Theorem 20.16.4 with $k_* = 35$. □

20.17 Integrated conditional closure theorem

Theorem 20.17.1 (Integrated conditional numerical closure)**.** *Fix* $\mu > 0$ *and a primitive sector assignment* $\sigma \mapsto k_\sigma \in \mathbb{N}$*. Suppose the following data are fixed:*

(i) *structural depth and carrier data from Theorem 20.7.1;*

(ii) *optional anchor/depth calibration route: either explicit anchors Theorem 20.5.1 or the ratio/cubic route Theorem 20.14.5;*

(iii) *optional cubic selector package Theorems 20.6.2 and 20.6.6.*

Then:

(a) $m_\sigma = 4\mu\binom{k_\sigma}{4}$ *for every primitive sector;*

(b) *mass ordering follows depth ordering for all depths* ≥ 4*;*

(c) *all anchor ratios are given by binomial quotients* $\binom{k_i}{4}/\binom{k_j}{4}$*;*

(d) *if the cubic defect first vanishes at depth* k_**, then* k_* *is the first cubic-admissible generation depth;*

(e) *if either the explicit depth-anchor package of Theorem 20.5.1 is stipulated or Theorem 20.14.5 applies, then the concrete* $(6, 18, 35)$ *depth-anchor package is fixed.*

Proof. Part (a) is Theorem 20.4.4, using Theorem 20.4.3. Part (b) is Theorem 20.10.3. Part (c) is Theorem 20.14.2. Part (d) is Theorem 20.16.4. Part (e) is immediate from Theorems 20.5.1 and 20.14.5. All components are conditional exactly on the stated structural and calibration data. □

Corollary 20.17.2 (Operational checklist)**.** *To apply Chapter 20 numerically, it is sufficient to specify:*

(i) *the depth assignment* k_σ*;*

(ii) *the normalization constant* μ*;*

(iii) *optional anchor/depth calibration route: either explicit reference depths, or the ratio/cubic route of Theorem 20.14.5;*

(iv) *optional cubic selector data* $\Delta_3(k)$ *(used in particular for the ratio/cubic route).*

After these data are fixed, every Chapter 20 output is an explicit algebraic consequence.

Proof. This is a direct restatement of Theorem 20.17.1 and the formulas proved in sections 20.9, 20.14 and 20.16. □

Remark 20.17.3 (Why this chapter is conditionally complete). From the structural side, multilinear combinatorics on $\Lambda^2(\mathbb{Z}^k)$ fixes the exact quartic backbone $E_4(\omega_k) = 4\binom{k}{4}$, while the broader admissible-depth rigidity package fixes quartic growth and quartic leading-order behavior at the saved theorem strength. From the quantitative side, the remaining degrees of freedom are named explicitly as conditional boundary data, and the formulas attached to each chosen data package display those dependencies directly. This is the precise sense in which Chapter 20 is complete as a *conditional* numerical closure theorem.

20.18 Conclusion

The chapter now yields a full dependency-explicit numerical package with three layers:

(i) The quartic combinatorics are exact and internally complete: $E_4(\omega_k) = 4\binom{k}{4}$ (Theorems 20.3.6 and 20.8.6).

(ii) The induced depth polynomial has explicit discrete calculus, recurrence, monotonicity, convexity, and inversion theory (Theorems 20.9.5, 20.9.6 and 20.11.2).

(iii) The quartic mass-coordinate law is identified by the intrinsic bridge theorem: $m_\sigma = 4\mu\binom{k_\sigma}{4}$ (Theorems 20.4.3 and 20.4.4).

(iv) Within a fixed anchor package, anchor ratios are given exactly by binomial quotients (Theorems 20.14.2 and 20.14.4).

(v) The explicit $(6, 18, 35)$ anchor package can be recovered via ratio/cubic closure and therefore need not be stipulated separately when that route is available (Theorems 20.14.5 and 20.14.6).

(vi) The cubic selector admits symmetric-polynomial normal forms and a general minimal-depth theorem (Theorems 20.16.3 and 20.16.4).

(vii) The full package is assembled as one conditional closure theorem (Theorems 20.17.1 and 20.17.2).

Accordingly, Chapter 20 now aligns with the proof architecture used in the longer structural chapters: it provides an exact quartic combinatorial core, an explicit dependency ledger, and a conditional chain from stated boundary data to the corresponding numerical outputs.

What remains beyond this chapter is precisely the status of in-shell splitting and final normalization: whether an additional intrinsic shell invariant is needed, whether its coefficient is internally fixed or irreducible, and how absolute mass scale is anchored. That extension is isolated in chapter 21.

Chapter 21

Intrinsic Numerical Closure, Unified Scalar Channel, and Same–Channel Corrections

21.1 Purpose and logical status

This chapter closes the intrinsic numerical layer of the closed relational theory. Its role is to separate what transport-closed structure fixes internally from what remains a distinct realization choice.

No observer-side semantic input enters the core closure statements. In particular, no external empirical normalization is required to state the intrinsic scalar law itself.

The objectives are:

(i) classify intrinsic scalar observables carried by the stabilized quadratic sector;

(ii) identify the single normalization orbit of that scalar channel;

(iii) constrain admissible subleading scalar corrections;

(iv) separate intrinsic sector observables from downstream physical interpretation maps.

The chapter proceeds in two layers. First comes the forced intrinsic closure chain for scalar depth laws. Second comes a compatibility bridge that preserves legacy shell labels without introducing new observer-side data.

21.2 Structural inputs from Chapters 19–20

We use the stabilized quadratic carrier and depth architecture established in chapters 19 and 20:

$$\mathcal{K} \cong F^2/F^3, \qquad Q : \mathcal{K}_{\mathbb{R}} \to \mathbb{R}_{\geq 0}, \qquad k \in \mathbb{N},$$

together with the quartic depth polynomial

$$q(k) := 4\binom{k}{4}$$

from Theorem 20.9.1.

We write

$$\Sigma(k) := Q(\kappa_k)$$

for the intrinsic scalar profile indexed by depth.

Methodologically, each statement below is typed as either *intrinsic/forced* or *compatibility/conditional*. This mirrors the bookkeeping discipline used in chapter 20.

Proposition 21.2.1 (Inherited dependency role in intrinsic form)**.** *The unconditional intrinsic statements in this chapter use two saved upstream inputs:*

(i) *the Chapter 19 scalar-channel framework Theorems 15.2.2, 19.5.3 and 19.6.3;*

(ii) *the Chapter 20 quartic-backbone and rigidity-envelope package, read with its saved distinction between structural consequences and extra compatibility inputs.*

Exact identification of a chosen scalar profile with $q(k) = 4\binom{k}{4}$*, exact ratio formulas for that profile, and any observer-side or Hilbert-branch realization statement require additional compatibility or realization hypotheses beyond those unconditional inputs.*

Proof. Chapter 19 fixes the unique scalar channel up to positive normalization, the canonical factorization of sector scalars through that channel, and transport covariance. Chapter 20 then supplies the quartic combinatorial backbone $q(k) = 4\binom{k}{4}$ for the named quartic observable together with the general quartic-growth and quartic-leading-law consequences used for admissible scalar depth observables. Those inputs support the unconditional intrinsic statements of this chapter at exactly that strength. Whenever the chapter speaks about exact equality with the quartic backbone, exact ratio formulas for a chosen mass-coordinate presentation, or the later Hilbert-branch/operator route, those claims must be typed explicitly as compatibility-level or realization-level statements rather than folded into the unconditional intrinsic package. □

21.3 Intrinsic scalar closure chain

This section begins with the forced intrinsic consequences of the Chapter 19/20 package and then marks later compatibility or conditional refinements explicitly where they enter.

Proposition 21.3.1 (Scaling covariance)**.** *If* Σ *is admissible and* $\lambda > 0$*, then*

$$\Sigma_\lambda(k) := \lambda\Sigma(k)$$

is admissible.

Proof. Admissibility is stable under positive rescaling, since the scalar channel is unique only up to positive normalization (Theorems 15.2.2 and 19.11.2). □

Theorem 21.3.2 (Intrinsic scalar-channel closure)**.** *Every admissible intrinsic scalar profile along the depth architecture remains in the unique scalar channel fixed by Theorems 15.2.2 and 19.5.3. Moreover, its leading growth is quartic in the sense that* $\Sigma(k) = O(k^4)$*, and under the Chapter 20 rigidity hypotheses its universal leading law is quartic.*

Proof. Channel uniqueness is exactly the Chapter 19 scalar-channel statement. The quartic growth envelope and quartic leading law are the saved Chapter 20 consequences recorded after patch P-218; they do not by themselves identify the full profile with the exact quartic backbone. □

Corollary 21.3.3 (Conditional ratio rigidity for the quartic backbone)**.** *Assume the chosen intrinsic scalar profile is identified with the Chapter 20 quartic observable up to one normalization constant, so that* $\Sigma(k) = \lambda q(k)$ *for some* $\lambda > 0$*. Then for any admissible* k_1, k_2 *with* $q(k_2) \neq 0$*,*

$$\frac{\Sigma(k_1)}{\Sigma(k_2)} = \frac{q(k_1)}{q(k_2)}.$$

Proof. Under the stated identification, the common factor λ cancels. □

Theorem 21.3.4 (Uniqueness of scalar channel)**.** *All intrinsic scalar observables descending from the stabilized quadratic carrier factor through one and the same one-dimensional scalar channel.*

Proof. This is precisely the channel-uniqueness content of Theorems 15.2.2 and 19.5.3. □

Theorem 21.3.5 (No absolute normalization)**.** *No intrinsic construction in Chapters 1–21 fixes the overall normalization constant* λ*.*

Proof. By Theorem 19.11.2, intrinsic structure fixes ratios, whereas absolute scale requires additional realization data. □

21.4 Correction structure inside the same channel

With the scalar channel fixed, we now record the compatibility model used later when one discusses subleading depth dependence without creating a new invariant type.

Theorem 21.4.1 (No new scalar invariants)**.** *Admissible scalar corrections do not create an independent scalar channel; they remain same-channel deformations of the intrinsic quartic profile.*

Proof. By Theorem 21.3.4, the scalar carrier is one-dimensional. Hence admissible scalar corrections may change depth dependence, but not the underlying scalar-channel type. □

Theorem 21.4.2 (Conditional binomial correction model). *Assume a same-channel scalar profile $M(k)$ is written as*

$$M(k) = \lambda\, 4\binom{k}{4} + R_{\le 2}(k),$$

where $\lambda \in \mathbb{R}$ and the remainder $R_{\le 2}(k)$ is a discrete polynomial of degree at most 2. Then there exist constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$M(k) = \alpha\binom{k}{4} + \beta\binom{k}{2} + \gamma k + \delta.$$

Proof. Because $R_{\le 2}(k)$ has degree at most 2, it can be written in the binomial basis $\beta\binom{k}{2} + \gamma k + \delta$. Absorbing the quartic backbone coefficient into $\alpha := 4\lambda$ yields the displayed form. □

Proposition 21.4.3 (Conditional leading structure). *Under the hypotheses of Theorem 21.4.2,*

$$M(k) = \alpha\binom{k}{4} + O(k^2) \qquad (k \to \infty).$$

In particular, the quartic backbone controls the leading asymptotic behavior.

Proof. The remainder terms $\beta\binom{k}{2} + \gamma k + \delta$ have degree at most 2, so they contribute only $O(k^2)$. □

Theorem 21.4.4 (Conditional cubic rigidity under the correction model). *Under the hypotheses of Theorem 21.4.2, the cubic coefficient of $M(k)$ is fixed entirely by the quartic backbone and cannot be independently deformed by the modeled remainder.*

Proof. The quartic backbone $4\binom{k}{4}$ already determines its own cubic term. Since the modeled remainder has degree at most 2, it cannot change that cubic coefficient. □

Proposition 21.4.5 (Conditional structure of modeled corrections). *Under the hypotheses of Theorem 21.4.2, modeled same-channel corrections first appear at quadratic order and below.*

Proof. This is exactly the degree statement encoded in the assumption on $R_{\le 2}(k)$ and restated by Theorem 21.4.4. □

21.5 Intrinsic spectral mass law, ratio rigidity, and asymptotic exactness

This section rewrites the intrinsic scalar closure chain in spectral-mass language and isolates the asymptotic structure of realization-level corrections. At intrinsic level, no new input is added beyond the scalar-channel and quartic-depth package Theorems 15.2.2, 19.5.3, 19.11.2, 20.4.2 and 20.9.1. Its statements are explicitly split into two types:

(i) *intrinsic/forced*: the one-channel structural facts, quartic leading behavior, and absence of intrinsic absolute normalization already fixed by Theorems 21.3.2 and 21.3.5;

(ii) *compatibility/conditional*: exact quartic-backbone formulas for a chosen mass-coordinate presentation, together with any consequences that require additional Hilbert-branch realization input beyond Chapters 1–21, namely the operator-route package Theorems 21.5.2, 21.5.4, 22.8.2, 22.8.4 and 22.8.6.

Define the quartic backbone

$$B_4(k) := 4\binom{k}{4} = \frac{k(k-1)(k-2)(k-3)}{6}, \qquad k \in \mathbb{N}.$$

By Theorem 20.9.1, this is exactly the intrinsic depth law $q(k)$.

Definition 21.5.1 (Intrinsic mass-coordinate profile)**.** Write

$$m_{\text{int}}(k) := \Sigma(k).$$

This is an intrinsic scalar profile (mass-coordinate language), not yet an observer-side measured mass assignment. Equivalently, it is the depth-indexed presentation of the same canonical scalar channel fixed by Theorems 15.2.2 and 19.5.3.

Theorem 21.5.2 (Conditional intrinsic spectral mass law)**.** *Assume the mass-coordinate profile is the Chapter 20 quartic backbone written in scalar-channel units, namely*

$$m_{\text{int}}(k) = \lambda\, 4\binom{k}{4}$$

for some $\lambda > 0$. Then the displayed quartic formulas, exact table, and exact ratio laws below hold for that chosen compatibility model.

Proof. This is a direct restatement of the assumed identification with the exact quartic backbone $q(k) = 4\binom{k}{4}$. □

Proposition 21.5.3 (Conditional unnormalized intrinsic mass table)**.** *Under Theorem 21.5.2, the quartic backbone and mass-coordinate profile are given by the displayed table for $k = 0, \dots, 20$:*

k	$B_4(k)$	$m_{\text{int}}(k)$
0	0	0
1	0	0
2	0	0
3	0	0
4	4	4λ
5	20	20λ
6	60	60λ
7	140	140λ
8	280	280λ
9	504	504λ
10	840	840λ
11	1320	1320λ
12	1980	1980λ
13	2860	2860λ
14	4004	4004λ
15	5460	5460λ
16	7280	7280λ
17	9520	9520λ
18	12240	12240λ
19	15504	15504λ
20	19380	19380λ

Proof. Direct evaluation of $B_4(k) = 4\binom{k}{4}$, followed by $m_{\text{int}}(k) = \lambda B_4(k)$ from Theorem 21.5.2. ☐

Theorem 21.5.4 (Conditional exact ratio law)**.** *Under Theorem 21.5.2, for any depths k, ℓ with $B_4(\ell) \neq 0$,*

$$\frac{m_{\text{int}}(k)}{m_{\text{int}}(\ell)} = \frac{B_4(k)}{B_4(\ell)} = \frac{\binom{k}{4}}{\binom{\ell}{4}}.$$

Proof. Immediate from Theorem 21.5.2. ☐

Corollary 21.5.5 (Conditional sample exact ratio predictions)**.** *Under Theorem 21.5.2, the displayed sample quartic-backbone ratios follow:*

$$\frac{m_{\text{int}}(5)}{m_{\text{int}}(4)} = 5, \qquad \frac{m_{\text{int}}(6)}{m_{\text{int}}(4)} = 15, \qquad \frac{m_{\text{int}}(18)}{m_{\text{int}}(6)} = 204, \qquad \frac{m_{\text{int}}(35)}{m_{\text{int}}(6)} = \frac{10472}{3}.$$

Proof. Substitute into Theorem 21.5.4. ☐

Theorem 21.5.6 (Normalization obstruction)**.** *The closed relational system does not determine the absolute normalization scale of m_{int}.*

Proof. This is exactly Theorem 21.3.5: intrinsic structure fixes only dimensionless ratios (Theorem 19.11.2) and leaves absolute normalization to separate realization data (Theorem 20.4.5 and section A.2). □

Corollary 21.5.7 (Intrinsic prediction content at saved strength)**.** *At unconditional intrinsic level, the theory determines the unique scalar channel, quartic leading behavior, and the absence of intrinsic absolute normalization. If one additionally identifies the mass-coordinate profile with the Chapter 20 quartic backbone up to normalization, then the exact quartic formulas and exact ratio consequences above follow.*

Proof. The unconditional intrinsic claims are exactly the one-channel and normalization statements from Theorems 21.3.2 and 21.3.5. The conditional quartic-backbone claims are Theorems 21.5.2 and 21.5.4. □

Remark 21.5.8 (Operator status and realization boundary). The operator statement needed to realize $m_0(\sigma) = Q(\kappa_\sigma)$ as Hilbert-branch spectral data is the target operator-route package Theorems 22.8.2, 22.8.4 and 22.8.6. Within Chapters 1–21, this remains a realization-level step rather than an already completed intrinsic theorem.

Definition 21.5.9 (Depth operator (conditional realization datum))**.** Assume a realized Hilbert branch H_{quad} with a depth-indexed sector family $\{\psi_k\}_{k\in\mathbb{N}} \subset H_{\text{quad}}$, each $\psi_k \neq 0$, compatible with the intrinsic depth label k from Theorem 19.9.1. Define the depth operator K on the finite span $D_{\text{fin}} := \text{span}\{\psi_k\}$ by

$$\mathsf{K}\psi_k = k\,\psi_k.$$

Definition 21.5.10 (Quartic backbone operator (conditional))**.** On D_{fin}, define

$$B_4(\mathsf{K}) := 4\binom{\mathsf{K}}{4} = \frac{\mathsf{K}(\mathsf{K}-1)(\mathsf{K}-2)(\mathsf{K}-3)}{6}.$$

Definition 21.5.11 (Realization defect operator (conditional model))**.** Assume a realized sector operator S_{real} on D_{fin} admits the decomposition

$$\mathsf{S}_{\text{real}} = \lambda B_4(\mathsf{K}) + R_{\text{def}},$$

where R_{def} is the realization defect operator.

Proposition 21.5.12 (Realized mass decomposition)**.** *Under Theorem 21.5.11, for every depth vector $\psi_k \in D_{\text{fin}}$,*

$$m_{\text{real}}(k) := \langle \psi_k, \mathsf{S}_{\text{real}}\psi_k\rangle = \lambda B_4(k) + \langle \psi_k, R_{\text{def}}\psi_k\rangle.$$

Proof. Insert $\mathsf{S}_{\text{real}} = \lambda B_4(\mathsf{K}) + R_{\text{def}}$ and use $B_4(\mathsf{K})\psi_k = B_4(k)\psi_k$. □

Definition 21.5.13 (Depth renormalization parameter)**.** Assume $B_4'(k) \neq 0$. Define

$$\delta_k := \frac{\langle \psi_k, R_{\text{def}}\psi_k\rangle}{\lambda B_4'(k)}.$$

Proposition 21.5.14 (First-order corrected spectrum)**.** *Under Theorem 21.5.13,*

$$m_{\text{real}}(k) = \lambda\big(B_4(k) + B_4'(k)\delta_k\big),$$

and hence

$$m_{\text{real}}(k) = \lambda B_4(k + \delta_k) + O\big(\delta_k^2 B_4''(k)\big)$$

to first order in realization defect.

Proof. By Theorem 21.5.13, $\langle \psi_k, R_{\text{def}}\psi_k\rangle = \lambda B_4'(k)\delta_k$. Substitute into Theorem 21.5.12. Then apply Taylor expansion of the quartic polynomial B_4. □

Lemma 21.5.15 (Quartic derivative asymptotics)**.** *For*

$$B_4(k) = 4\binom{k}{4} = \frac{k(k-1)(k-2)(k-3)}{6},$$

one has

$$B_4'(k) = \frac{4k^3 - 18k^2 + 22k - 6}{6} = \frac{2}{3}\,k^3 + O(k^2),$$

and

$$B_4''(k) = \frac{12k^2 - 36k + 22}{6} = 2k^2 + O(k).$$

Proof. Differentiate the polynomial expression for $B_4(k)$ directly. □

Theorem 21.5.16 (Asymptotic exactness of the quartic mass law)**.** *Assume*

$$\langle \psi_k, R_{\text{def}}\psi_k\rangle = O(k^p) \qquad \text{for some } p < 3.$$

Then:

(i)
$$\delta_k = O(k^{p-3}), \qquad \delta_k \to 0 \quad (k \to \infty);$$

(ii)
$$m_{\text{real}}(k) = \lambda B_4(k) + o(B_4(k));$$

(iii)
$$\frac{m_{\text{real}}(k)}{\lambda B_4(k)} \to 1 \qquad (k \to \infty).$$

In particular, if $\langle \psi_k, R_{\text{def}}\psi_k\rangle = O(k^2)$*, then* $\delta_k = O(k^{-1})$*.*

Proof. By Theorem 21.5.13,

$$\delta_k = \frac{\langle \psi_k, R_{\text{def}}\psi_k\rangle}{\lambda B_4'(k)}.$$

By hypothesis, $\langle \psi_k, R_{\text{def}}\psi_k\rangle = O(k^p)$, and by Theorem 21.5.15, $B_4'(k) = \Theta(k^3)$. Hence $\delta_k = O(k^{p-3})$, so $\delta_k \to 0$ because $p < 3$.

Next,

$$m_{\text{real}}(k) = \lambda B_4(k) + \langle \psi_k, R_{\text{def}}\psi_k \rangle = \lambda B_4(k) + O(k^p).$$

Since $B_4(k) = \frac{1}{6}k^4 + O(k^3) = \Theta(k^4)$ and $p < 3 < 4$, one has $O(k^p) = o(B_4(k))$. Therefore $m_{\text{real}}(k) = \lambda B_4(k) + o(B_4(k))$, and dividing by $\lambda B_4(k)$ gives item (iii). The final sentence is the case $p = 2$. ☐

Proposition 21.5.17 (Conditional exact-ratio test for a chosen depth assignment)**.** *Assume the chosen mass-coordinate profile is identified with the quartic backbone as in Theorem 21.5.2. For any proposed depth assignment*

$$k_1, \ldots, k_n,$$

that compatibility model then predicts exact ratios

$$\frac{m_{\text{int}}(k_i)}{m_{\text{int}}(k_j)} = \frac{B_4(k_i)}{B_4(k_j)}.$$

Hence the quartic-backbone compatibility model admits an exact unnormalized ratio test.

Proof. Immediate from Theorem 21.5.4. ☐

Remark 21.5.18 (Interpretive consequence). Under the same quartic-backbone identification, if observed realized ratios for a chosen depth assignment are close to the exact quartic ratios, this supports using that quartic backbone as the leading intrinsic model for the chosen assignment. Residual discrepancy is then naturally modeled as lower-order realization defect, and under Theorem 21.5.16 this defect is asymptotically negligible.

Theorem 21.5.19 (Complete intrinsic spectral prediction at the saved split)**.** *The closed relational system determines the scalar channel and its quartic leading behavior, but not an intrinsic absolute normalization. Under the explicit quartic-backbone identification used in this section, one further obtains the exact table and exact ratio formulas for that compatibility model. Under an additional subcubic realization-defect bound, the realized spectrum is asymptotically exact relative to that quartic backbone in the sense of Theorem 21.5.16.*

Proof. The unconditional intrinsic statement is exactly the one-channel and quartic-leading-behavior package summarized in Theorems 21.3.2 and 21.3.5. The conditional quartic-backbone table and ratio formulas are Theorems 21.5.2 to 21.5.4. The final statement is Theorem 21.5.16. ☐

21.6 Handoff to matter-sector realization

This chapter closes intrinsic scalar closure at the level of quartic leading behavior and a single normalization orbit, while retaining same-channel correction structure only as a compatibility bridge for modeled same-channel corrections. The next chapter (chapter 22) handles branch-level realization of the canonical sector invariant $m_0(\sigma) = Q(\kappa_\sigma)$.

Its first remaining theorem target is the Hilbert-branch operator realization Theorem 22.8.2, via the closed positive form route. It then isolates the explicit boundary between intrinsic sector observables and observer-side interpretation maps.

21.6.1 Forced-upgrade roadmap for the operator target

A full promotion of the Chapter 22 operator target Theorem 22.8.2 from realization/conditional status to intrinsic/forced status requires a theorem chain proved from existing theorem-level data only. The following labeled items record the minimal roadmap targets whose later discharge would suffice for that upgrade: canonical sector-to-Hilbert embedding, induced sector form construction, and closed-form closability.

Theorem 21.6.1 (Roadmap target: canonical sector-to-Hilbert embedding)**.** *To promote the Chapter 22 operator statement to intrinsic/forced status, it would suffice to prove the following statement from only Theorems 13.10.2, 19.4.1, 19.6.1 and 19.9.1 and chapter 15: there exists a canonical linear map*

$$\iota_{\mathrm{sec}} : \mathrm{span}\{\kappa_\sigma : \sigma \textit{ primitive excitation sector}\} \hookrightarrow H_{\mathrm{quad}}$$

with $\iota_{\mathrm{sec}}(\kappa_\sigma) = \psi_\sigma \neq 0$, compatible with intrinsic depth labels and transport covariance.

Lemma 21.6.2 (Roadmap target: embedded sector form realizes intrinsic scalar values)**.** *Assuming Theorem 21.6.1, the later intrinsic upgrade would also be secured by proving the following target statement from Theorems 15.2.2, 15.2.6 and 19.5.3: define on the finite sector span*

$$q_{\mathrm{sec}}(\psi, \phi) := \left\langle \iota_{\mathrm{sec}}^{-1}\psi, \iota_{\mathrm{sec}}^{-1}\phi \right\rangle_Q .$$

Then q_{sec} is positive and densely defined, and for each primitive sector σ,

$$q_{\mathrm{sec}}(\psi_\sigma, \psi_\sigma) = m_0(\sigma)\, \|\psi_\sigma\|^2.$$

Hence q_{sec} would be of the input type required by Theorem 22.8.4.

Theorem 21.6.3 (Roadmap target: closed-form closability from intrinsic control)**.** *Assuming Theorem 21.6.2, the later intrinsic upgrade would further be secured by proving this target statement: if intrinsic Hilbert-branch control estimates imply that q_{sec} is closable, then its closure $\overline{q}_{\mathrm{sec}}$ is a closed positive form on H_{quad}. This is exactly the closed/closable-form hypothesis needed in Theorem 22.8.6.*

Proposition 21.6.4 (Roadmap implication for a later forced upgrade of the Chapter 22 operator target)**.** *If Theorems 21.6.1 to 21.6.3 were later discharged from existing theorem-level inputs alone, then that discharge would supply the sufficiency route through Theorem 22.8.6 for promoting Theorem 22.8.2 to intrinsic/forced status.*

Proof. A later proof of the first two roadmap targets would provide the sector-form data required by Theorem 22.8.4. A later proof of the third would provide the closed/closable-form hypothesis used by Theorem 22.8.6. Together, those later discharges would therefore yield the canonical positive self-adjoint operator with sector eigenvalue law $\mathsf{S}\psi_\sigma = m_0(\sigma)\psi_\sigma$, which is exactly the content that would promote Theorem 22.8.2 to intrinsic/forced status. □

Corollary 21.6.5 (Forced-upgrade criterion recorded at roadmap strength). *A full promotion from realization/conditional to intrinsic/forced for the Chapter 22 operator statement is justified exactly when the roadmap targets Theorems 21.6.1 to 21.6.3 are later proved without adding new observer-side assumptions.*

Proof. Immediate from Theorem 21.6.4 and section A.2. □

21.7 Compatibility bridge for legacy shell labels

For consistency with the existing ledger and cross-chapter references, we record a compact compatibility layer in legacy shell notation. These labels are optional interface markers and do not add observer-side inputs to the intrinsic core established above. They function as interface-level translators between the current closure language and older shell bookkeeping.

Definition 21.7.1 (Signed shell profile (compatibility form)). At fixed depth, a signed shell profile is a finite family $(\Delta_\eta(\sigma))_{\eta\in\Lambda_\sigma}$ with $\Delta_\eta(\sigma) \in \mathbb{Z}$.

Definition 21.7.2 (Shell splitting invariant (compatibility form)). Define

$$S(\sigma) := \sum_{\eta\in\Lambda_\sigma} \Delta_\eta(\sigma)^2.$$

Definition 21.7.3 (Shell completeness property (legacy label)). A realized branch is *shell-complete* at fixed depth when shell-visible scalar distinction factors through $S(\sigma)$.

Theorem 21.7.4 (Signed-permutation criterion for shell completeness). *If the shell residual at depth k is quadratic and invariant under independent sign flips and coordinate permutations of shell coordinates, then it is proportional to $\sum_\eta \Delta_\eta^2$. If the proportionality coefficient is depth-independent, then the shell-completeness property of Theorem 21.7.3 holds.*

Proof. The invariance forces diagonal isotropy of the associated symmetric bilinear form, so the quadratic residual is a scalar multiple of the Euclidean square sum. Depth-independent proportionality then yields a single universal shell channel. □

Corollary 21.7.5 (Conditional discharge of shell completeness). *Whenever Theorem 21.7.4 holds with depth-independent coefficient, the shell-completeness property Theorem 21.7.3 need not be stipulated independently.*

Proof. Immediate from Theorem 21.7.4. □

Theorem 21.7.6 (Transport-to-shell derivation of symmetry hypotheses)**.** *Fix depth k and an interface map $I_k : \mathbb{R}^{\Lambda_k} \to \mathcal{K}_{\mathbb{R}}$. If*

$$R_k(\Delta) = Q(I_k\Delta)$$

and for each signed permutation g there is a realized transport action T_g on $\mathcal{K}_{\mathbb{R}}$ with

$$I_k(g\Delta) = T_g I_k(\Delta), \qquad Q(T_g x) = Q(x),$$

then R_k satisfies the symmetry hypotheses of Theorem 21.7.4.

Proof. Linearity of I_k and quadraticity of Q make R_k quadratic. Intertwining together with Q-invariance implies invariance of R_k under the signed-permutation action. Hence the symmetry hypotheses are satisfied. □

Theorem 21.7.7 (Interface checklist for transport-derived symmetry)**.** *To verify condition* (iii) *of Theorem 21.7.6 in-model, it suffices to check intertwining and Q-invariance on finite generators: coordinate sign flips s_i and adjacent transpositions τ_i, together with their signed-permutation relations.*

Proof. These generators and relations present the hyperoctahedral group. If intertwining and Q-invariance hold on generators, they extend to all group elements by composition. □

Corollary 21.7.8 (Transport-derived shell-completeness route)**.** *If Theorem 21.7.6 holds for each depth and the induced proportionality coefficients are depth-independent, then the shell-completeness property Theorem 21.7.3 and Theorem 21.7.5 hold.*

Proof. Combine Theorems 21.7.4 to 21.7.6. □

Definition 21.7.9 (One-reference defect-evaluation datum (legacy label))**.** A one-reference defect-evaluation datum is a non-balanced reference sector with intrinsic defect data sufficient to recover the dimensionless shell ratio parameter in one-reference form.

Definition 21.7.10 (One-reference anchor datum (legacy label))**.** A one-reference anchor datum is a reference sector with fixed intrinsic scalar value chosen for absolute normalization in one-reference form.

Theorem 21.7.11 (Two-reference algebraic closure)**.** *Assume a shell-corrected normalized law*

$$m(\sigma) = \alpha\big(q(k_\sigma) + \gamma S(\sigma)\big)$$

in a realized branch. For two reference sectors σ_1, σ_2 with known m_i, k_i, S_i $(i = 1, 2)$, if $m_1S_2 - m_2S_1 \neq 0$, then

$$\gamma = \frac{m_2q(k_1) - m_1q(k_2)}{m_1S_2 - m_2S_1},$$

and

$$\alpha = \frac{m_1}{q(k_1) + \gamma S_1} = \frac{m_2}{q(k_2) + \gamma S_2}.$$

Proof. Write the normalized law at σ_1, σ_2, eliminate α by cross-multiplication, solve for γ, and then substitute back to recover α. □

Corollary 21.7.12 (Discharge of one-reference closure data)**.** *Under Theorem 21.7.11, the one-reference data Theorems 21.7.9 and 21.7.10 are optional.*

Proof. Two-reference algebraic recovery determines both the dimensionless parameter and the absolute normalization directly. □

21.8 Final closure statement

Theorem 21.8.1 (Maximal intrinsic closure)**.** *The closed system determines:*

(i) *quartic structural depth law up to one normalization orbit;*

(ii) *ratio rigidity and absence of intrinsic absolute normalization.*

In addition, under the chapter's explicit same-channel correction model, the cubic coefficient is fixed by the quartic backbone and the modeled remainder begins at quadratic order. No additional intrinsic scalar degree of freedom is left implicit.

Proof. The intrinsic claims follow from Theorems 19.11.2, 21.3.2 and 21.3.5. The modeled correction statement follows from Theorems 21.4.1, 21.4.2, 21.4.4 and 21.4.5 under the explicit ansatz used in section 21.4. □

21.9 Conclusion

The intrinsic numerical sector is now closed at theorem level: the scalar law is fixed up to one normalization orbit, and any later same-channel correction discussion remains an explicit compatibility model rather than a new intrinsic scalar channel.

Accordingly, this chapter closes the intrinsic one-channel numerical core of the main manuscript at saved strength: quartic leading behavior and normalization-orbit freedom are fixed, while same-channel correction models and realization-level identifications remain explicit rather than forced. The branch-level realization role of the canonical matter-sector scalar is taken up next in chapter 22.

Chapter 22

Matter-Sector Realization of the Canonical Scalar Invariant

22.1 Purpose and logical status

Chapter 21 closes the intrinsic numerical classification of the scalar channel at the level of quartic leading behavior and a single normalization orbit, while recording same-channel correction structure only as compatibility-level bookkeeping. What remains is not further intrinsic classification, but realization role. For the Chapter 21 mass-coordinate reformulation and the accompanying conditional exact-ratio package used here as downstream numerical bookkeeping, see section 21.5.

This chapter identifies that role for the canonical excitation-sector scalar invariant. No observer-side measured-mass interpretation is assumed here. The objective is purely internal: determine which already-established realization branch canonically carries the Chapter 19 sector invariant.

The argument unfolds in four steps:

(i) type the intrinsic scalar datum and its realization roles;

(ii) prove branch priority and the intrinsic/observer boundary;

(iii) isolate the operator-realization gap and its form route;

(iv) formalize observer-map factorization constraints.

22.2 Structural inputs

We use the following theorem-level inputs from earlier chapters:

(i) stabilized quadratic carrier $\mathcal{K} \cong F^2/F^3$ (Theorem 13.10.2);

(ii) one-dimensional canonical scalar channel on $\mathcal{K}_{\mathbb{R}}$ (Theorems 15.2.2 and 19.5.3);

(iii) Hilbert realization branch of the stabilized quadratic carrier (chapter 15);

(iv) phase realization branch of the same carrier (chapter 18);

(v) macroscopic Einstein-branch scalar coefficient, read under the inherited second-jet faithfulness condition on the realized degree–2 channel attached to the stabilized quadratic carrier (chapter 16);

(vi) canonical sector defect classes and scalar invariants $\kappa_\sigma \in \mathcal{K}$, $m_0(\sigma) := Q(\kappa_\sigma)$ (Theorems 19.4.1, 19.6.1 and 19.9.1).

The sections below keep these inputs separated by role: carrier typing, branch realization, and observer-side factorization.

22.3 Canonical sector scalar

Definition 22.3.1 (Canonical sector scalar)**.** For a primitive excitation sector σ, define

$$m_0(\sigma) := Q(\kappa_\sigma),$$

where $\kappa_\sigma \in \mathcal{K}$ is the canonical quadratic defect class and Q is the unique admissible scalar channel on $\mathcal{K}_\mathbb{R}$.

Remark 22.3.2 (Chapter 19 role). By Theorems 19.6.2, 19.6.3 and 19.9.1, $m_0(\sigma)$ is strictly positive, transport-invariant, and part of the intrinsic sector-separation package.

22.4 Typing lemmas for realization roles

Lemma 22.4.1 (Carrier factorization of the canonical sector scalar)**.** *For every primitive excitation sector σ, the scalar $m_0(\sigma)$ factors canonically as*

$$\sigma \longmapsto \kappa_\sigma \in \mathcal{K} \hookrightarrow \mathcal{K}_\mathbb{R} \xrightarrow{Q} \mathbb{R}_{\geq 0}.$$

In particular, m_0 is intrinsic and transport-invariant before any observer-side interpretation map is introduced.

Proof. By Theorem 19.4.1, each primitive sector has a canonical defect class $\kappa_\sigma \in \mathcal{K}$. By Theorem 15.2.2, the scalar channel is a canonical positive quadratic map $Q : \mathcal{K}_\mathbb{R} \to \mathbb{R}_{\geq 0}$, and Theorem 19.6.1 defines $m_0(\sigma) = Q(\kappa_\sigma)$. Transport invariance is exactly Theorem 19.6.3, and strict positivity for primitive nontrivial sectors is Theorem 19.6.2. Hence the displayed factorization and intrinsic status are fixed prior to any external semantics. □

Proposition 22.4.2 (Microscopic/macroscopic role split on one scalar channel)**.** *The canonical scalar channel admits two theorem-level realization roles:*

(i) *microscopic excitation-sector role, where* $m_0(\sigma) = Q(\kappa_\sigma)$ *participates in primitive-sector classification;*

(ii) *macroscopic geometric role, where, under the inherited second-jet faithfulness condition, the same scalar channel is read as the Einstein-branch scalar coefficient on the realized degree–2 channel attached to the stabilized quadratic carrier.*

These roles are type-distinct and are not identified with one another without an additional realization map.

Proof. For (i), Chapter 19 defines m_0 on primitive sectors and uses it inside sector-separation structure (Theorems 19.6.1 and 19.9.1). For (ii), Chapter 16 shows that, under the inherited second-jet faithfulness condition, the macroscopic scalar coefficient is read from the same canonical degree–2 scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier, up to normalization (Theorems 16.8.1 and 16.9.5).

The codomains and logical roles differ: (i) is a scalar invariant attached to primitive excitation sectors, while (ii) is a coefficient in the macroscopic geometric operator classification. Therefore no theorem-level identification between the two roles follows without extra data. □

22.5 Branch selection

Theorem 22.5.1 (Matter-sector branch selection)**.** *Let* σ *be a primitive excitation sector and* $m_0(\sigma) = Q(\kappa_\sigma)$ *its canonical scalar invariant. Then the admissible first realization role of* $m_0(\sigma)$ *is through the Hilbert–phase branch of the stabilized quadratic carrier. Equivalently:*

(i) *as a sector observable,* $m_0(\sigma)$ *is carried by the microscopic quadratic Hilbert law and its phase descent;*

(ii) $m_0(\sigma)$ *is not, by default, a matter-sector observable in the macroscopic Einstein branch, where, under the inherited second-jet faithfulness condition, the same scalar channel is read as the Einstein-branch scalar coefficient on the realized degree–2 channel attached to the stabilized quadratic carrier.*

Proof. By Theorem 22.4.1, $m_0(\sigma)$ is attached to the stabilized quadratic defect class $\kappa_\sigma \in \mathcal{K}$. The first realization branch of this carrier is the microscopic Hilbert-sector realization (Theorem 15.2.2) together with its phase descent (Theorems 18.4.1 and 18.9.1). So the intrinsic excitation-sector scalar is already realized on the Hilbert–phase branch.

By Theorem 22.4.2, under the inherited second-jet faithfulness condition the Einstein branch reads the same scalar channel as macroscopic geometric coefficient data on the realized degree–2 channel attached to the stabilized quadratic carrier, rather than as primitive-sector observable data. Since Chapter 19 employs $m_0(\sigma)$ as part of primitive excitation-sector organization (Theorem 19.9.1), its first admissible realization role is the microscopic Hilbert–phase role. Hence (i) and (ii) follow. □

22.6 Sector-observable priority

Theorem 22.6.1 (Sector-observable priority)**.** *For every primitive excitation sector σ,*

$$m_0(\sigma) = Q(\kappa_\sigma)$$

is, prior to any observer-side semantics, an intrinsic sector observable.

Proof. By Theorem 22.4.1, the map $\sigma \mapsto m_0(\sigma)$ is defined purely from $\kappa_\sigma \in \mathcal{K}$ and the intrinsic scalar channel Q, and is transport-invariant before any observer semantics. Chapter 19 then uses this intrinsic scalar inside sector-organization data (Theorem 19.9.1). By Theorem 22.5.1, this scalar is first realized on the Hilbert–phase branch at sector level. Therefore its status as a sector observable is fixed internally, prior to any observer-side interpretation. □

22.7 Realization boundary

Corollary 22.7.1 (Mass-realization boundary)**.** *Any identification of the intrinsic sector scalar $m_0(\sigma)$, or of the Chapter 20 closed-system mass coordinate, with an observer-side measured mass observable requires a separate semantic realization theorem. It is not part of the intrinsic theorem package of chapters 14 to 19 and 21.*

Proof. By Theorem 22.6.1, $m_0(\sigma)$ is theorem-level intrinsic sector data. By Theorem 22.5.1 and Theorem 22.4.2, this intrinsic datum has a canonical microscopic role and is not automatically identified with the Einstein-branch scalar coefficient read, under the inherited second-jet faithfulness condition, on the realized degree–2 channel attached to the stabilized quadratic carrier.

Hence any external measured-mass interpretation must introduce an additional semantic readout map, schematically

$$I_{\text{mass}} : m_0(\sigma) \longmapsto m_{\text{obs}}(\sigma),$$

not determined by the intrinsic package listed in the statement. Chapter 20 proves the internal quartic mass-coordinate bridge (Theorems 20.4.3 and 20.4.5); what is separate is the observer-side identification of that internal coordinate with a measured mass convention. Therefore external measured-mass interpretation remains a semantic realization theorem, not an internal axiom. □

22.8 Operator-first spectral realization gap

The next theorem-level strengthening is not orthogonality as a starting input. The exact remaining direction is to construct the Chapter 19 sector scalar invariant as a self-adjoint operator on the Hilbert realization branch. Orthogonality is then a consequence of spectral structure.

From the existing chapter stack, we have the following proved data:

- stabilized quadratic carrier $\mathcal{K} \cong F^2/F^3$ (Theorem 13.10.2);
- unique one-dimensional scalar channel on $\mathcal{K}_{\mathbb{R}}$ (Theorems 15.2.2 and 19.5.3);
- Hilbert realization of the quadratic carrier (chapter 15);
- intrinsic excitation-sector data κ_σ, $m_0(\sigma) = Q(\kappa_\sigma)$, k_σ, which separates primitive sectors inside a fixed gauge class (Theorem 19.9.1).

None of these, by itself, yields a canonical densely defined positive self-adjoint operator on the Hilbert realization whose sector eigenvalues are the intrinsic scalars $m_0(\sigma)$.

Proposition 22.8.1 (Nature of the realization gap)**.** *Intrinsic sector separation does not by itself produce the Hilbert-branch sector scalar operator.*

Proof. Sector separation is proved at classification level in the stabilized quadratic carrier, through invariants such as κ_σ, $m_0(\sigma)$, and k_σ (Theorems 19.4.1, 19.6.1 and 19.9.1). Operator realization is a statement about unbounded self-adjoint structure on the Hilbert branch. No theorem in the current chain yet constructs the canonical positive self-adjoint operator whose sector eigenvalues are the intrinsic scalars. Hence the remaining gap is an operator-construction gap. □

Theorem 22.8.2 (Target operator realization of the sector scalar invariant)**.** *Let H_{quad} be the Hilbert realization of the stabilized quadratic carrier $\mathcal{K} \simeq F^2/F^3$ from Chapter 15. The exact remaining realization-level operator target is to prove that there exists a canonical densely defined positive self-adjoint operator*

$$\mathsf{S} : D(\mathsf{S}) \subset H_{\mathrm{quad}} \to H_{\mathrm{quad}}$$

such that for every primitive excitation sector σ one has a nonzero vector $\psi_\sigma \in D(\mathsf{S})$ satisfying

$$\mathsf{S}\psi_\sigma = m_0(\sigma)\psi_\sigma, \qquad m_0(\sigma) = Q(\kappa_\sigma).$$

Remark 22.8.3 (Status of the target theorem). Theorem 22.8.2 records the exact remaining realization-level operator statement to be discharged. It is not part of the current intrinsic theorem package proved so far.

Lemma 22.8.4 (Target form-typing input for the sector operator route)**.** *A sufficient intermediate target for Theorem 22.8.2 is to prove that there exists a densely defined positive sesquilinear form*

$$\mathfrak{q} : D(\mathfrak{q}) \times D(\mathfrak{q}) \to \mathbb{C}$$

on H_{quad} such that for every primitive excitation sector σ,

$$\mathfrak{q}(\psi_\sigma, \psi_\sigma) = m_0(\sigma)\|\psi_\sigma\|^2.$$

Remark 22.8.5 (Conditional candidate for the form route). If the later sector-to-Hilbert embedding and form-typing targets are discharged, then the natural candidate for the form in Theorem 22.8.4 is the defect-carrier expression

$$\mathfrak{q}(\psi, \phi) := \langle \kappa(\psi), \kappa(\phi) \rangle_Q,$$

where $\kappa(\psi)$ denotes the quadratic defect representative attached to the embedded sector vector ψ, and $\langle \cdot, \cdot \rangle_Q$ is the polarized form induced by the scalar channel Q (Theorem 15.2.6) together with its later sesquilinear Hilbert-branch extension. This remark records only that candidate route; it does not assert that the required embedding, form-typing, or closability input has already been proved.

Proposition 22.8.6 (Closed-form route to the sector operator)**.** *Assume Theorem 22.8.4 and assume the form* $\mathfrak{q}$ *is closed or closable. Then the representation theorem for closed positive forms yields a canonical positive self-adjoint operator*

$$\mathsf{S} : D(\mathsf{S}) \subset H_{\text{quad}} \to H_{\text{quad}}$$

satisfying

$$\mathfrak{q}(\psi, \phi) = \langle \psi, \mathsf{S}\phi \rangle$$

for all $\psi \in D(\mathfrak{q})$, $\phi \in D(\mathsf{S})$.

Proof. If $\mathfrak{q}$ is closed and positive, the standard representation theorem assigns a unique positive self-adjoint operator S whose form domain is $D(\mathfrak{q})$ and whose form identity is exactly the displayed equality. If $\mathfrak{q}$ is closable, apply the same theorem to its closure. Canonicity follows from uniqueness in the representation theorem. □

22.8.1 Proof checklist for the forced-upgrade roadmap

This subsection maps the roadmap stubs of section 21.6.1 to exact proof slots inside the Chapter 22 operator route.

Proposition 22.8.7 (Later roadmap-target bookkeeping for the Chapter 22 operator route)**.** *If the later roadmap targets recorded in section 21.6.1 were discharged, then they would feed this chapter as follows:*

(i) *Theorem 21.6.1 would supply the canonical sector-to-Hilbert embedding input needed to place sector vectors* $\psi_\sigma \in H_{\text{quad}}$ *in the operator-construction route of section 22.8;*

(ii) *Theorem 21.6.2 would supply the form-typing input slot formalized by Theorem 22.8.4;*

(iii) *Theorem 21.6.3 would discharge the closed/closable hypothesis required by Theorem 22.8.6.*

Proof. Immediate from the repaired target status of Theorems 22.8.4 and 22.8.6 and the later-upgrade bookkeeping already recorded in section 21.6.1 and Theorem 21.6.4. □

Corollary 22.8.8 (Later checklist discharge promotes the operator target)**.** *If the three roadmap targets named in Theorem 22.8.7 are later proved from existing theorem-level inputs without new observer-side assumptions, then that later discharge promotes the Chapter 22 operator target Theorem 22.8.2 through the closed-form route.*

Proof. Combine Theorems 21.6.4, 22.8.6 and 22.8.7. □

Lemma 22.8.9 (Sector orthogonality from operator realization)**.** *Let* S *be self-adjoint on* H_{quad}*, and suppose*

$$\mathsf{S}\psi_\sigma = m_0(\sigma)\psi_\sigma, \qquad \mathsf{S}\psi_\tau = m_0(\tau)\psi_\tau.$$

Then

$$m_0(\sigma) \neq m_0(\tau) \Longrightarrow \langle \psi_\sigma, \psi_\tau \rangle = 0.$$

Proof. Self-adjointness gives

$$\langle \mathsf{S}\psi_\sigma, \psi_\tau \rangle = \langle \psi_\sigma, \mathsf{S}\psi_\tau \rangle.$$

Hence

$$m_0(\sigma)\langle \psi_\sigma, \psi_\tau \rangle = m_0(\tau)\langle \psi_\sigma, \psi_\tau \rangle,$$

so

$$(m_0(\sigma) - m_0(\tau))\langle \psi_\sigma, \psi_\tau \rangle = 0.$$

If $m_0(\sigma) \neq m_0(\tau)$, then $\langle \psi_\sigma, \psi_\tau \rangle = 0$. □

Corollary 22.8.10 (Consequences conditional on sector-observable realization)**.** *Assume Theorem 22.8.2 and*

$$H_{\mathrm{quad}} = \overline{\mathrm{span}\{\psi_\sigma : \sigma \in \mathsf{S}_{\mathrm{prim}}\}}.$$

Then:

(i) $m_0(\sigma) \neq m_0(\tau) \Rightarrow \langle \psi_\sigma, \psi_\tau \rangle = 0$*;*

(ii) *the sector eigenvectors generate a genuine spectral family for* S*;*

(iii) *canonical spectral projections for* S *exist and determine the sector decomposition of the Hilbert branch.*

Proof. Item (i) is Theorem 22.8.9. Items (ii) and (iii) are standard consequences of the spectral theorem for self-adjoint operators once the eigenvector family is complete in the stated closure. □

In this way, orthogonality, spectral families, and projections appear as consequences of operator realization, not as independent starting input. The remaining realization step is therefore exact and isolated: construct the sector operator by closed positive form methods.

22.9 Observer-map factorization algebra

Write

$$\mathsf{S}_{\text{prim}} := \{\sigma : \sigma \text{ primitive excitation sector}\}, \qquad m_0 : \mathsf{S}_{\text{prim}} \to \mathbb{R}_{\geq 0},\ \sigma \mapsto Q(\kappa_\sigma).$$

The goal of this section is formal and minimal: isolate exactly when an observer-side assignment descends through the intrinsic scalar channel.

Definition 22.9.1 (Observer-side mass assignment). An observer-side mass assignment is any map

$$M_{\text{obs}} : \mathsf{S}_{\text{prim}} \to \mathbb{R}_{>0}.$$

Theorem 22.9.2 (Factorization criterion through the intrinsic scalar). *Let M_{obs} be an observer-side mass assignment. The following are equivalent:*

(i) *there exists a map $\Lambda : \operatorname{Im}(m_0) \to \mathbb{R}_{>0}$ such that*

$$M_{\text{obs}} = \Lambda \circ m_0;$$

(ii) *for all $\sigma_1, \sigma_2 \in \mathsf{S}_{\text{prim}}$,*

$$m_0(\sigma_1) = m_0(\sigma_2) \Longrightarrow M_{\text{obs}}(\sigma_1) = M_{\text{obs}}(\sigma_2).$$

When these conditions hold, Λ is unique.

Proof. (i)⇒(ii): if $M_{\text{obs}} = \Lambda \circ m_0$ and $m_0(\sigma_1) = m_0(\sigma_2)$, then

$$M_{\text{obs}}(\sigma_1) = \Lambda(m_0(\sigma_1)) = \Lambda(m_0(\sigma_2)) = M_{\text{obs}}(\sigma_2).$$

(ii)⇒(i): define $\Lambda : \operatorname{Im}(m_0) \to \mathbb{R}_{>0}$ by

$$\Lambda(x) := M_{\text{obs}}(\sigma) \quad \text{for any } \sigma \text{ with } m_0(\sigma) = x.$$

Condition (ii) makes this well-defined. Then for every σ,

$$(\Lambda \circ m_0)(\sigma) = \Lambda(m_0(\sigma)) = M_{\text{obs}}(\sigma),$$

so $M_{\text{obs}} = \Lambda \circ m_0$. For uniqueness, if $\Lambda_1 \circ m_0 = \Lambda_2 \circ m_0$, then for every $x \in \operatorname{Im}(m_0)$ choose σ with $m_0(\sigma) = x$, giving $\Lambda_1(x) = M_{\text{obs}}(\sigma) = \Lambda_2(x)$. Hence $\Lambda_1 = \Lambda_2$. □

Corollary 22.9.3 (Non-injectivity obstruction). *Assume there exist $\sigma_1 \neq \sigma_2$ with*

$$m_0(\sigma_1) = m_0(\sigma_2), \qquad M_{\text{obs}}(\sigma_1) \neq M_{\text{obs}}(\sigma_2).$$

Then M_{obs} cannot factor through m_0. Hence any such assignment requires additional sector observables beyond the intrinsic scalar channel.

Proof. If M_{obs} factored through m_0, Theorem 22.9.2 would force equality $M_{\text{obs}}(\sigma_1) = M_{\text{obs}}(\sigma_2)$, contradicting the hypothesis. The final statement is immediate. □

Proposition 22.9.4 (Two-point affine calibration on $\mathrm{Im}(m_0)$)**.** *Fix two distinct intrinsic scalar values $x_1, x_2 \in \mathrm{Im}(m_0)$ and target values $y_1, y_2 \in \mathbb{R}$. There is a unique affine map $\Lambda_{\text{aff}}(x) = ax + b$ satisfying*

$$\Lambda_{\text{aff}}(x_1) = y_1, \qquad \Lambda_{\text{aff}}(x_2) = y_2,$$

namely

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \qquad b = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}.$$

Proof. The constraints are the linear system

$$ax_1 + b = y_1, \qquad ax_2 + b = y_2.$$

Subtract to obtain $a(x_2 - x_1) = y_2 - y_1$, hence the stated formula for a. Substituting into either equation gives the stated formula for b. Because $x_2 \neq x_1$, the system has unique solution. □

Remark 22.9.5 (Boundary-data interpretation). Theorem 22.9.2 and Theorem 22.9.4 formalize the same logical boundary recorded in Theorem 22.7.1: the intrinsic sector scalar defines the factorization channel, while the choice of observer map Λ (or calibration parameters) is additional realization data.

22.10 Conclusion

The canonical scalar invariant of Chapter 19 is now placed at its proper realization level. It is:

- intrinsic;
- sector-theoretic;
- first realized through the Hilbert–phase branch;
- constrained by an explicit observer-map factorization criterion;
- not yet an observer-side mass observable without extra calibration.

The remaining intrinsic realization step is explicit and isolated: construct the Hilbert-branch sector operator by the closed positive form route recorded in Theorems 22.8.2, 22.8.4 and 22.8.6. Any later physical-mass identification would still require a separate observer-side realization theorem or calibration input beyond that operator construction.

Part VIII

Appendix

Appendix A

Conditional Completion Ledger

A.1 Extracted structural content

This ledger records entries that are structurally reduced but not yet numerically fixed by closed transport alone. In the Chapter 20/21 terminology, such entries are called *extracted*. No additional theorem-level claim is introduced here; this chapter is status bookkeeping for downstream numerical closure.

At the current manuscript stage, the extracted entry is:

(i) full representation-content classification reduced to irreducible unitary representation data of $U(1) \times SU(2) \times SU(3)$ (Theorems 19.2.7 and 19.2.8).

A.2 Open quantitative boundary data

This section records quantitatively open entries that are not determined by the theorem-level closure chain of Chapters 1–21. Unless further closure theorems are proved, these entries remain explicit boundary data.

After the global closure audit of Chapters 14–21, the ledger is typed in three layers.

Layer A (internal bridge data discharged). The scalar bilinear source is forced by polarization (Theorem 14.8.2), and the closed-system quartic mass coordinate is fixed by the intrinsic readout bridge (Theorem 20.4.3).

Layer B (properties and data discharged by theorem). The explicit depth-anchor package Theorem 20.5.1 is optional: it is discharged whenever the ratio/cubic calibration criterion of Theorems 20.14.5 and 20.14.6 is available.

The shell-completeness property Theorem 21.7.3 is optional: it is discharged whenever the signed-permutation quadratic criterion Theorems 21.7.4 and 21.7.5 is available, in particular via the Chapter 19 transport-derived route Theorems 21.7.6 and 21.7.8, with finite in-model verification by Theorem 21.7.7.

The one-reference shell closure data Theorems 21.7.9 and 21.7.10 are no longer mandatory: they are discharged whenever the two-reference algebraic closure criterion of Theorems 21.7.11 and 21.7.12 is available.

Layer C (residual open quantitative-status entries).

(i) explicit real-level form of the scalar quadratic channel $Q_{\mathbb{R}}$ (Theorems 19.5.1 and 19.5.4);

(ii) shell-completeness status (explicit property declaration vs verified signed-permutation discharge criterion, including the Chapter 19 transport-derived route) for data packages where Theorems 21.7.4, 21.7.6 and 21.7.7 is not established;

(iii) dimensionless shell-splitting constant status (internal determination vs irreducible γ) when neither the intrinsic-defect route nor two-reference algebraic closure is available;

(iv) absolute mass-scale normalization (open α) when no calibration route fixes α;

(v) generation-level realization details;

(vi) low-energy coupling-constant values.

These entries remain outside current theorem-level closure and are tracked as explicit open boundary data.

A.3 Formal hardening ledger

This section lists every global item that carries non-theorem-level load and records its strongest current status. The words "datum", "property", and "principle" are used deliberately: a datum is retained only as boundary data, a property may be verified or discharged theoremically, and the standing principle is the admissibility rule that governs the whole construction.

(H1) **Closed-world admissibility.** Standing Principle 1 is the sole standing principle. It is not a hidden geometric, probabilistic, or dynamical postulate; it is the rule that only internally reconstructed comparison content is admissible.

(H2) **Scalar pairing bridge.** Theorem 14.8.2 proves that the bilinear source needed to pass from the canonical scalar channel to a real quadratic form is forced by polarization; it is no longer retained as boundary data.

(H3) **Closed-system mass-coordinate bridge.** Theorem 20.4.3 proves that the intrinsic quartic coefficient-energy readout is the closed-system mass coordinate. Only the external identification of that internal coordinate with a measured mass convention lies outside the internal theorem stack.

(H4) **Depth-anchor package.** Theorem 20.5.1 is a package, not a condition. It is stipulated directly only for data packages that choose explicit $(6, 18, 35)$ reference depths, and it is recovered by the ratio/cubic route Theorems 20.14.5 and 20.14.6.

(H5) **Shell completeness.** Theorem 21.7.3 is a property, not a condition. It is proved by the signed-permutation quadratic criterion and, in transport-realized settings, by the transport-to-shell route Theorems 21.7.4, 21.7.6 and 21.7.8.

(H6) **One-reference defect data.** Theorem 21.7.9 is one-reference shell bookkeeping data only. It is unnecessary whenever the two-reference algebraic closure theorem applies.

(H7) **One-reference anchor data.** Theorem 21.7.10 is likewise one-reference normalization data only. It is discharged together with defect-evaluation data by Theorems 21.7.11 and 21.7.12.

Definition A.3.1 (Global closure-status vector)**.** The current global closure-status vector is

$$\mathcal{C}_{\text{status}} := (P, B, M, A, S, R, E),$$

where:

(i) P is the standing closed-world admissibility principle Standing Principle 1;

(ii) B is the scalar-pairing bridge Theorem 14.8.2;

(iii) M is the closed-system mass-coordinate bridge Theorem 20.4.3;

(iv) A is the depth-anchor package, either specified by Theorem 20.5.1 or recovered by Theorem 20.14.5;

(v) S is shell completeness, either verified by Theorems 21.7.4 and 21.7.6 or retained as an open property;

(vi) R is one-reference shell bookkeeping, discharged when Theorem 21.7.11 applies;

(vii) E is the residual observer-side semantic realization layer.

Theorem A.3.2 (Global closure audit normal form)**.** *At the present theorem strength, every global non-structural dependency in Chapters 14–22 is represented by exactly one component of* $\mathcal{C}_{\text{status}}$. *Moreover:*

(i) P *is the only standing principle;*

(ii) B *and* M *are theorem-level bridges, not boundary data;*

(iii) A, S, *and* R *are either specified packages/properties or are discharged by the theorems named in Theorem A.3.1;*

(iv) *the only item not internalized by the mathematical closure stack is* E*, the observer-side interpretation of closed-system coordinates as empirical readouts.*

Proof. The standing layer is unique by Standing Principle 1 and by the introductory scope statement, so all global admissibility load is accounted for by P.

For scalar structure, Theorem 14.8.2 constructs the bilinear source by polarization from the real quadratic scalar channel. Thus the old pairing-level load is represented by B and is theorem-level. For mass coordinates, Theorem 20.4.3 identifies the internal quartic coefficient-energy readout with the closed-system mass coordinate; therefore the corresponding load is represented by M and is also theorem-level.

For numerical anchors, Theorem 20.5.1 records the explicit package, while Theorems 20.14.5 and 20.14.6 recover and discharge it when the ratio/cubic route applies. This is exactly component A. For shell completeness, Theorem 21.7.3 names the property and Theorems 21.7.4, 21.7.6 and 21.7.8 give the theorem-level verification routes; this is exactly component S. For one-reference shell bookkeeping, Theorems 21.7.9 and 21.7.10 name the data and Theorems 21.7.11 and 21.7.12 discharge them under two-reference closure; this is exactly component R.

The residual entries listed in section A.2 either refine one of A, S, or R, or concern observer-side empirical interpretation. Theorems 22.7.1 and 22.9.2 isolate that interpretation as a separate semantic realization problem, which is component E. Hence every global non-structural dependency is represented by one listed component, and the four claims follow. □

Corollary A.3.3 (No hidden global assumptions after hardening)**.** *After replacing each occurrence of old assumption-level language by the corresponding component of* $\mathcal{C}_{\text{status}}$*, the manuscript has no unlisted global quantitative assumption in Chapters 14–22.*

Proof. By Theorem A.3.2, every global non-structural dependency belongs to exactly one component of $\mathcal{C}_{\text{status}}$. Components B and M are theorems; components A, S, and R are either named explicitly or discharged by named theorems; component E is explicitly semantic rather than mathematical. Therefore no remaining global quantitative assumption is hidden. □

Remark A.3.4 (Irreducible residue at current theorem strength)*.* After this hardening, Theorems 20.5.1, 21.7.3, 21.7.9 and 21.7.10 carry no assumption-level status: each is either a defined package, a theorem-proved property, or optional one-reference bookkeeping discharged by a two-reference theorem. The scalar-pairing and closed-system mass-coordinate bridge theorems are now proved internally in Theorems 14.8.2 and 20.4.3. The only residue is semantic: an external observer may still ask whether the closed-system mass coordinate is the measured mass used in a chosen empirical protocol. That identification is not an additional mathematical axiom of the closed-system stack; it is an observer-side realization convention.

Appendix B

Nonperturbative Adiabatic Invariants from Closed-System Transport

B.1 Purpose and logical status

Under the standing principle of closed-world admissibility (Standing Principle 1), this appendix records a downstream confinement application of items (SP2) and (SP5) through the stabilized quadratic scalar channel. As the terminal step of the manuscript, it isolates the theorem-level intrinsic scalar carried by any confinement system realized as a closed comparison world and then separates that scalar from its later perturbative and magnetic-confinement readings. The purpose of this appendix is to present a downstream application template for the closed relational stack developed in the main text. The chosen downstream model is magnetic confinement. The conclusion at theorem level is that, once the transport stack and the inherited realized scalar-channel package of Theorems 13.10.2, 14.7.6 and 15.2.2 are in force, one obtains a scalar

$$\mathfrak{s}_{\mathrm{fus}} : U \to \mathbb{R}$$

which is unique up to nonzero normalization and invariant under the intrinsic evolution of the confinement system. This appendix is entirely downstream. It introduces no new foundational principle, assumes no new dynamical axiom locally, and does not reprove the closed-system stack. Its role is to show that once the classification, quotient, transport, quadratic-carrier, and realized scalar-channel theorems are available, the intrinsic scalar

$$\mathfrak{s}_{\mathrm{fus}} = Q \circ \kappa$$

follows formally. The operative chain is:

$$\begin{aligned}(U,\mathcal{C}) &\Longrightarrow U \cong X_A \times X_B\\ &\Longrightarrow \pi : X \to \mathsf{Phys} = X/G\\ &\Longrightarrow F^1 \supset F^2 \supset F^3 \supset \cdots\\ &\Longrightarrow \mathcal{K}_\infty \subset F^2/F^3\\ &\Longrightarrow Q : \mathcal{K}_\infty \to \mathbb{R}\\ &\Longrightarrow \mathfrak{s}_{\text{fus}} = Q \circ \kappa.\end{aligned}$$

The point is exact at the level of the intrinsic scalar. The quadratic scalar channel isolated in the main text is therefore not left as a formal carrier datum. In the confinement setting it furnishes a canonical nonperturbative transport scalar; later perturbative matching and physical interpretation are treated separately below.

Theorem B.1.1 (Main appendix theorem)**.** *Under the standing principle Standing Principle 1, let* $(U,\mathcal{C})$ *be the closed comparison world associated to a magnetic confinement system. Assume the structural inputs listed in section B.2. Then there exists a scalar-valued function*

$$\mathfrak{s}_{\text{fus}} : U \to \mathbb{R}$$

such that:

(a) $\mathfrak{s}_{\text{fus}}$ *is preserved by every intrinsic evolution*

$$\Phi_t \in \mathrm{Aut}(U,\mathcal{C}),$$

that is,

$$\mathfrak{s}_{\text{fus}}(\Phi_t(u)) = \mathfrak{s}_{\text{fus}}(u) \qquad \textit{for all } u \in U,\ t;$$

(b) $\mathfrak{s}_{\text{fus}}$ *is unique up to multiplication by a nonzero real scalar.*

Proof. Existence is proved in Theorem B.5.3, invariance under intrinsic evolution in Theorem B.5.4, and uniqueness up to nonzero scaling in Theorem B.5.5. Combining these three statements yields items (a)–(b). □

B.2 Structural inputs from the main text

The appendix uses the transport/carrier results already established in the main text together with the realized scalar-channel package already isolated there. No new primitive axiom is introduced locally. The required inputs are:

(i) the closed-world quotient backbone and structural completion of admissible state content (*Theorem* 2.10.1);

(ii) the lift and transport-obstruction architecture, including the canonical morphism-locus carrier and its first visible quadratic layer (*Theorems* 6.6.1*and* 12.5.5);

(iii) stabilization of the intrinsic obstruction channel and its realization interface $\big(Theorem\ 13.10.2\big)$;

(iv) existence and one-dimensional rigidity of the scalar channel on the realized degree–2 channel attached to the stabilized quadratic carrier, together with its probabilistic Born normalization under the inherited second-jet interface hypothesis $\big(Theorems\ 14.7.6and\ 15.2.2\big)$.

These inputs are exactly the data propagated in the remainder of the appendix to produce the intrinsic confinement scalar $\mathfrak{s}_{\mathrm{fus}} = Q \circ \kappa$. Any perturbative identification or magnetic-confinement reading is kept as a downstream interpretation layer on top of that scalar.

B.3 Transport filtration and the universal quadratic quotient

Let

$$N := \mathrm{Ker}\big(\mathrm{Aut}(U, \mathcal{C}) \to \mathrm{Aut}(X_A) \times \mathrm{Aut}(X_B)\big), \qquad F^1 := N, \qquad F^{m+1} := [F^1, F^m].$$

Definition B.3.1 (Visible degree–1 transport module)**.**

$$V := F^1/F^2.$$

Commutator structure of the transport filtration

Lemma B.3.2. *For all integers $i, j \geq 1$,*

$$[F^i, F^j] \subseteq F^{i+j}.$$

In particular, each quotient F^m/F^{m+1} is abelian.

Proof. We proceed in a sequence of logically independent steps. **Step 1: Reduction to a universal group-theoretic statement.** The filtration is defined inductively by

$$F^{m+1} = [F^1, F^m].$$

Thus $F^\bullet$ is the lower central series of the group F^1. It suffices to prove that for any group G with lower central series

$$\gamma_1(G) = G, \qquad \gamma_{m+1}(G) = [G, \gamma_m(G)],$$

one has

$$[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G).$$

We therefore prove the statement in this generality. **Step 2: Induction scheme.**

Define

$$P(n): \quad [\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G) \quad \text{for all } i+j \leq n.$$

We prove $P(n)$ for all $n \geq 2$ by induction. **Step 3: Base case.** For $n = 2$, the only possibility is $i = j = 1$. Then

$$[\gamma_1(G), \gamma_1(G)] = [G, G] = \gamma_2(G),$$

so the statement holds. **Step 4: Inductive step.** Assume $P(n-1)$. Let $i + j = n$. Without loss of generality, assume $i \geq 2$. Then

$$\gamma_i(G) = [G, \gamma_{i-1}(G)].$$

Thus

$$[\gamma_i(G), \gamma_j(G)] = [[G, \gamma_{i-1}(G)], \gamma_j(G)].$$

We now invoke the Hall–Witt identity (three-subgroup identity): for subgroups A, B, C,

$$[[A, B], C] \subseteq [A, [B, C]] \cdot [B, [A, C]].$$

Apply with

$$A = G, \quad B = \gamma_{i-1}(G), \quad C = \gamma_j(G).$$

Then

$$[\gamma_i(G), \gamma_j(G)] \subseteq [G, [\gamma_{i-1}, \gamma_j]] \cdot [\gamma_{i-1}, [G, \gamma_j]].$$

By the induction hypothesis,

$$[\gamma_{i-1}, \gamma_j] \subseteq \gamma_{i+j-1}(G),$$

and by definition,

$$[G, \gamma_j] = \gamma_{j+1}(G).$$

Therefore,

$$[G, [\gamma_{i-1}, \gamma_j]] \subseteq [G, \gamma_{i+j-1}] = \gamma_{i+j}(G),$$

and

$$[\gamma_{i-1}, [G, \gamma_j]] = [\gamma_{i-1}, \gamma_{j+1}] \subseteq \gamma_{(i-1)+(j+1)}(G) = \gamma_{i+j}(G).$$

Thus

$$[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G).$$

Step 5: Abelianity of graded quotients. Taking $i = j = m$,

$$[\gamma_m(G), \gamma_m(G)] \subseteq \gamma_{2m}(G).$$

Since $2m \geq m + 1$,

$$\gamma_{2m}(G) \subseteq \gamma_{m+1}(G),$$

hence

$$[\gamma_m(G), \gamma_m(G)] \subseteq \gamma_{m+1}(G).$$

Therefore the quotient

$$\gamma_m(G)/\gamma_{m+1}(G)$$

is abelian. **Step 6: Specialization.** Applying this to $G = F^1$, we obtain

$$[F^i, F^j] \subseteq F^{i+j},$$

as required. □

Universal quadratic quotient

Theorem B.3.3 (Universal quadratic quotient)**.** *There exists a canonical surjective homomorphism*

$$q_\wedge : \Lambda^2(V) \twoheadrightarrow F^2/F^3$$

characterized by

$$q_\wedge(\bar{x} \wedge \bar{y}) = [x, y] \bmod F^3.$$

Proof. We divide the proof into five logically independent parts. **Step 1: Definition of the pairing.** Let $\bar{x}, \bar{y} \in V = F^1/F^2$. Choose representatives $x, y \in F^1$. Define

$$\beta(\bar{x}, \bar{y}) := [x, y] \bmod F^3.$$

We must show that β is:

1. well defined;
2. bilinear;
3. alternating;
4. universal;
5. surjective.

Step 2: Independence of representatives. Suppose

$$x' = xf, \quad y' = yg, \quad f, g \in F^2.$$

We compute

$$[x', y'] = [xf, yg].$$

Using the standard commutator expansion (valid in any group),

$$[ab, cd] = [a, d] \cdot [a, c]^d \cdot [b, d]^c \cdot [b, c],$$

and simplifying modulo F^3, we obtain the classical reduction:

$$[xf, yg] = [x, y] \cdot [x, g] \cdot [f, y] \cdot [f, g] \qquad \text{mod } F^3.$$

Now: - $[x, g] \in [F^1, F^2] = F^3$, - $[f, y] \in [F^2, F^1] = F^3$, - $[f, g] \in [F^2, F^2] \subseteq F^4 \subseteq F^3$. Hence

$$[xf, yg] \equiv [x, y] \pmod{F^3}.$$

Thus β is well defined. **Step 3: Bilinearity.** Let $\bar{x}, \bar{x}', \bar{y} \in V$, with lifts $x, x', y \in F^1$. We compute:

$$[xx', y] = [x, y] \cdot [x, y, x'] \cdot [x', y],$$

where

$$[x, y, x'] := ([[x, y], x']).$$

Since $[x, y] \in F^2$,

$$[x, y, x'] \in [F^2, F^1] = F^3.$$

Thus modulo F^3,

$$[xx', y] \equiv [x, y] \cdot [x', y].$$

Passing to the quotient F^2/F^3, multiplication becomes addition, hence

$$\beta(\bar{x} + \bar{x}', \bar{y}) = \beta(\bar{x}, \bar{y}) + \beta(\bar{x}', \bar{y}).$$

The same argument applies in the second variable. **Step 4: Alternation.** We have

$$[x, x] = e \Rightarrow \beta(\bar{x}, \bar{x}) = 0.$$

Also

$$[y, x] = [x, y]^{-1},$$

which corresponds to negation in the abelian group F^2/F^3. Thus β is alternating. **Step 5: Factorization.** By the universal property of the exterior square, every alternating bilinear map

$$V \times V \to F^2/F^3$$

factors uniquely through

$$\Lambda^2(V).$$

Thus there exists a unique homomorphism

$$q_\wedge : \Lambda^2(V) \to F^2/F^3$$

with

$$q_\wedge(\bar{x} \wedge \bar{y}) = \beta(\bar{x}, \bar{y}).$$

Step 6: Surjectivity. By definition,

$$F^2 = [F^1, F^1].$$

Thus every element of F^2 is a finite product of commutators:

$$z = \prod_i [x_i, y_i].$$

Passing to F^2/F^3, the group becomes abelian, so

$$z \equiv \sum_i [x_i, y_i].$$

Hence every element lies in the image of $q_\wedge$. **Conclusion.** Therefore $q_\wedge$ is a well-defined surjective homomorphism. □

Definition B.3.4 (Quadratic relation subgroup)**.** Define

$$R_2 := \mathrm{Ker}(q_\wedge) \subseteq \Lambda^2(V).$$

Then there is an exact sequence

$$0 \longrightarrow R_2 \longrightarrow \Lambda^2(V) \xrightarrow{q_\wedge} F^2/F^3 \longrightarrow 0.$$

Remark B.3.5. This identifies the universal degree–2 commutator object. The invariant theorem does not require solving the full relation problem on $\Lambda^2(V)$. By Theorem 13.10.7, every admissible degree–2 scalarization already vanishes on R_2, and therefore factors uniquely through the quadratic carrier. Thus the scalar theory of the appendix is intrinsic to the quadratic carrier F^2/F^3 and not to any choice of universal lift.

B.4 The stabilized quadratic carrier and its realized scalar channel

Section B.3 constructs the universal degree–2 quotient. We now pass to the stabilized carrier that actually controls the invariant.

Proposition B.4.1. *The compatible degree–2 transport defects define the stabilized quadratic carrier*

$$K = \mathcal{K}_\infty$$

of Theorem 13.10.2.

Proof. We make explicit the logical content of the cited theorem in the notation of the present appendix. By construction, the transport package of the main text assigns to each finite stage of the refinement tower a degree–2 carrier

$$\mathcal{K}_k \subseteq F^2 K^{(k)}_{p_k} / F^3 K^{(k)}_{p_k}$$

generated by the quadratic square classes arising from the intrinsic transport defect at that stage. The refinement morphisms induce compatible transition maps

$$\mathcal{K}_{k+1} \longrightarrow \mathcal{K}_k$$

because the defect construction commutes with refinement; this is the content of the compatibility statements proved upstream and assembled in Theorem 13.10.2. A compatible family

$$(\xi_k)_k, \qquad \xi_k \in \mathcal{K}_k,$$

is by definition an element of the inverse limit

$$\varprojlim_k \mathcal{K}_k.$$

Conversely, every element of the inverse limit is, by definition, such a compatible family. The cited theorem identifies this inverse-limit object as the stabilized degree–2 transport carrier and denotes it by

$$\mathcal{K}_\infty.$$

Therefore the stabilized carrier relevant for the present appendix is precisely

$$K := \mathcal{K}_\infty.$$

No additional structure is introduced here; the proposition is exactly the translation of Theorem 13.10.2 into the present notation. □

Proposition B.4.2. *Under the inherited second-jet faithfulness condition and compatible smooth realization, nonzero stabilized square classes in the stabilized quadratic carrier are equivalent to nonzero realized curvature on the realized degree–2 channel.*

Proof. The statement has two parts: first, the passage from intrinsic quadratic transport defect to the stabilized carrier; second, the realized interface criterion carried by compatible smooth realization. By Theorem 13.12.3, if

$$\mathcal{A}_{ij}(p_\infty) \in \mathcal{K}_\infty$$

denotes the stabilized quadratic square class determined by the (i, j)-transport square at the inverse-limit basepoint p_∞, then under the second-jet comparison map

$$\mathsf{Jet}_2 : \mathcal{K}_\infty \to \mathrm{End}(V), \qquad V := T_{\iota(p_\infty)}M,$$

nonzero stabilized square classes are equivalent to nonzero realized curvature on the realized degree–2 channel. By Theorem 17.9.1, the same compatible smooth realization supplies the realized affine-connection and metric branch on which this curvature is read. Thus the appendix inherits the repaired main-text interface exactly: the stabilized quadratic carrier feeds the confinement application through the inherited second-jet criterion relating its nonzero stabilized square classes to nonzero realized curvature, not through a direct whole-carrier curvature identity. □

Proposition B.4.3 (Realized scalar channel on the stabilized carrier)**.** *Under smooth realization and the inherited second-jet interface hypothesis, the stabilized quadratic carrier admits a nonzero scalar channel*

$$Q : K \to \mathbb{R}$$

read from the realized degree–2 channel attached to that carrier, unique up to multiplication by a nonzero real scalar.

Proof. By Theorem B.4.2, under smooth realization and the inherited second-jet faithfulness condition, the stabilized quadratic carrier

$$K = \mathcal{K}_\infty$$

is read through the realized curvature carrier. The scalarization problem used in this appendix is therefore exactly the scalarization problem on the realized degree–2 channel addressed in the main text. Now Theorems 14.7.4 and 14.7.5 state that the space of admissible scalarizations of that realized degree–2 channel is one-dimensional. Concretely, this means that if

$$W_1, W_2 : K \to \mathbb{R}$$

are admissible scalarizations, then there exists

$$c \in \mathbb{R}$$

such that

$$W_1 = c\, W_2.$$

In particular, the scalarization space is nonzero, so there exists at least one nontrivial scalarization. Fix any such nonzero scalarization and denote it by

$$Q : K \to \mathbb{R}.$$

If $W : K \to \mathbb{R}$ is any other admissible scalarization, then the one-dimensionality assertion implies that there exists

$$c \in \mathbb{R}$$

with

$$W = cQ.$$

If $W \neq 0$, then necessarily $c \neq 0$. Hence every nonzero admissible scalarization is a nonzero scalar multiple of Q. This is the scalar channel used in the remainder of the appendix. □

Canonical quadratic defect section

The construction of the confinement invariant uses a canonical map

$$\kappa : U \to K$$

from the state space to the stabilized quadratic carrier. To make the downstream scalarization completely rigid, we isolate the construction in three stages: existence, basepoint-independence, and functorial uniqueness.

Theorem B.4.4 (Canonical quadratic defect section)**.** *Under the standing principle Standing Principle 1, let* $(U, \mathcal{C})$ *be a closed comparison world, let*

$$G := \mathrm{Aut}(U, \mathcal{C}),$$

and let

$$K = \mathcal{K}_\infty$$

be the stabilized quadratic transport carrier obtained from the refinement-compatible degree–2 transport system of chapter 13. In the local witness-selection setup, assume that for each state $u \in U$*, the refinement tower admits a compatible family of local transport witnesses centered at* u*, so that at each finite stage* k *one has a local witness network*

$$\mathcal{N}_k(u) = (V_k(u), E_k(u)),$$

a chosen admissible basepoint

$$p_k(u) \in V_k(u),$$

and associated quadratic square classes

$$\mathcal{A}_{ij}^{(k)}(p_k(u)) \in F^2 K_{p_k(u)}^{(k)} / F^3 K_{p_k(u)}^{(k)}.$$

Assume moreover that these classes are compatible under the refinement bonding maps. Then the family

$$\left(\mathcal{A}_{ij}^{(k)}(p_k(u))\right)_k$$

defines a canonical element

$$\kappa(u) \in K.$$

Hence there exists a canonical map

$$\kappa : U \to K, \qquad u \longmapsto \kappa(u),$$

called the quadratic defect section. Furthermore, for every

$$g \in G \qquad \text{and} \qquad u \in U,$$

one has

$$\kappa(g \cdot u) = g_* \kappa(u),$$

where

$$g_* : K \to K$$

is the induced automorphism of the stabilized quadratic carrier.

Proof. We divide the argument into six steps. **Step 1: Finite-stage quadratic defect classes attached to a state.** Fix

$$u \in U.$$

By hypothesis, at each refinement stage k, the state u determines a local transport witness system with basepoint

$$p_k(u) \in V_k(u),$$

and therefore a based defect homomorphism

$$\delta^{(k)}_{p_k(u)} : \Omega_{p_k(u)}(\mathcal{N}_k(u)) \to K^{(k)}_{p_k(u)}.$$

For each ordered pair of local transport directions (i, j), let

$$\square_{ij} := e_i e_j e_i^{-1} e_j^{-1}$$

be the corresponding commutator square loop based at $p_k(u)$. By the quadratic cancellation result upstream, the defect element

$$\delta^{(k)}_{p_k(u)}(\square_{ij})$$

lies in

$$F^2 K^{(k)}_{p_k(u)}.$$

Its class modulo F^3 is therefore well defined:

$$\mathcal{A}^{(k)}_{ij}(p_k(u)) := \left[\delta^{(k)}_{p_k(u)}(\square_{ij})\right] \in F^2 K^{(k)}_{p_k(u)} / F^3 K^{(k)}_{p_k(u)}.$$

Thus each state u determines, stage by stage, a family of finite-stage quadratic transport defect classes. **Step 2: Compatibility under refinement.** Let

$$R^{\Omega}_{k+1,k} : \Omega_{p_{k+1}(u)}(\mathcal{N}_{k+1}(u)) \to \Omega_{p_k(u)}(\mathcal{N}_k(u))$$

and

$$Q^K_{k+1,k} : K^{(k+1)}_{p_{k+1}(u)} \to K^{(k)}_{p_k(u)}$$

be the bonding maps supplied by refinement compatibility. By the compatibility identity for defect maps,

$$Q^K_{k+1,k} \circ \delta^{(k+1)}_{p_{k+1}(u)} = \delta^{(k)}_{p_k(u)} \circ R^{\Omega}_{k+1,k},$$

the defect of the commutator square at stage $k+1$ maps to the defect of the corresponding commutator square at stage k. Passing to the quadratic quotient yields

$$Q^{(2)}_{k+1,k}\big(\mathcal{A}^{(k+1)}_{ij}(p_{k+1}(u))\big) = \mathcal{A}^{(k)}_{ij}(p_k(u)),$$

where

$$Q^{(2)}_{k+1,k} : F^2 K^{(k+1)}_{p_{k+1}(u)} / F^3 K^{(k+1)}_{p_{k+1}(u)} \to F^2 K^{(k)}_{p_k(u)} / F^3 K^{(k)}_{p_k(u)}$$

is the induced degree–2 bonding morphism. Therefore

$$\big(\mathcal{A}_{ij}^{(k)}(p_k(u))\big)_k$$

is a compatible family in the inverse system defining the stabilized carrier. **Step 3: Passage to the inverse limit.** By definition,

$$K = \mathcal{K}_\infty = \varprojlim_k \mathcal{K}_k.$$

Hence every compatible family of finite-stage quadratic classes defines a unique element of K. Applying this to the family attached to u, we obtain an element

$$\kappa(u) \in K.$$

This constructs a map

$$\kappa : U \to K, \qquad u \mapsto \kappa(u).$$

At this stage the map is defined relative to the chosen basepoint system $(p_k(u))_k$. The next theorem removes that dependence. **Step 4: Finite-stage equivariance under intrinsic automorphisms.** Let

$$g \in G = \mathrm{Aut}(U, \mathcal{C}).$$

Because g preserves the comparison world, it preserves the induced transport structure. In particular, for each k, it carries local witness systems at u to local witness systems at $g \cdot u$, sends commutator squares to commutator squares, and induces an isomorphism

$$g_*^{(k)} : F^2 K_{p_k(u)}^{(k)} / F^3 K_{p_k(u)}^{(k)} \to F^2 K_{p_k(g \cdot u)}^{(k)} / F^3 K_{p_k(g \cdot u)}^{(k)}$$

satisfying

$$g_*^{(k)}\big(\mathcal{A}_{ij}^{(k)}(p_k(u))\big) = \mathcal{A}_{ij}^{(k)}(p_k(g \cdot u)).$$

Since the family $(g_*^{(k)})_k$ is compatible with refinement, it assembles into an induced automorphism

$$g_* : K \to K.$$

Step 5: Naturality. The k-th component of $\kappa(u)$ is

$$\mathcal{A}_{ij}^{(k)}(p_k(u)).$$

Applying g_* to $\kappa(u)$ acts componentwise by $g_*^{(k)}$, so the k-th component of $g_*\kappa(u)$ is

$$g_*^{(k)}\big(\mathcal{A}_{ij}^{(k)}(p_k(u))\big) = \mathcal{A}_{ij}^{(k)}(p_k(g \cdot u)).$$

But this is exactly the k-th component of $\kappa(g \cdot u)$. Since equality in an inverse limit is equivalent to equality of all finite-stage components, it follows that

$$\kappa(g \cdot u) = g_*\kappa(u).$$

Step 6: Conclusion. Thus the map

$$\kappa : U \to K$$

is canonically defined from compatible quadratic defect classes and is natural under the intrinsic symmetry action. □

Theorem B.4.5 (Basepoint-independence of the quadratic defect section)**.** *Let the hypotheses of Theorem B.4.4 hold. Fix a state*

$$u \in U.$$

At each finite refinement stage k, let

$$p_k,\ p'_k \in V_k(u)$$

be two admissible basepoints representing the same state u, and let

$$\gamma_k : p_k \to p'_k$$

be an admissible transport path between them. Let

$$\mathcal{A}^{(k)}_{ij}(p_k), \qquad \mathcal{A}^{(k)}_{ij}(p'_k)$$

be the quadratic defect classes computed using these two basepoints. Then

$$\mathcal{A}^{(k)}_{ij}(p'_k) = \mathcal{A}^{(k)}_{ij}(p_k) \quad in \quad F^2K^{(k)}/F^3K^{(k)}.$$

Consequently, the element

$$\kappa(u) \in K$$

is independent of the choice of basepoint representatives.

Proof. We prove the theorem in four steps. **Step 1: Conjugation identifies based loop groups.** Fix a stage k. Let

$$\gamma_k : p_k \to p'_k$$

be an admissible transport path. Then conjugation by γ_k identifies the based loop groups:

$$\Omega_{p_k}(\mathcal{N}_k(u)) \longrightarrow \Omega_{p'_k}(\mathcal{N}_k(u)), \qquad \ell \longmapsto \gamma_k\, \ell\, \gamma_k^{-1}.$$

Under the transport functor, this induces conjugation on the isotropy group:

$$\delta^{(k)}_{p'_k}(\gamma_k \ell \gamma_k^{-1}) = \delta^{(k)}_{\gamma_k}\, \delta^{(k)}_{p_k}(\ell)\, \delta^{(k)\,-1}_{\gamma_k},$$

where

$$\delta^{(k)}_{\gamma_k} \in K^{(k)}$$

denotes the transport element associated to γ_k. **Step 2: The two commutator-square defects are conjugate.** Let

$$\square_{ij} := e_i e_j e_i^{-1} e_j^{-1}$$

be the commutator square based at p_k, and let

$$\square'_{ij} := \gamma_k \, \square_{ij} \, \gamma_k^{-1}$$

be the corresponding square at p'_k. Then

$$\delta^{(k)}_{p'_k}(\square'_{ij}) = \delta^{(k)}_{\gamma_k} \, \delta^{(k)}_{p_k}(\square_{ij}) \, \delta^{(k)\,-1}_{\gamma_k}.$$

Set

$$x := \delta^{(k)}_{p_k}(\square_{ij}) \in F^2 K^{(k)}, \qquad g := \delta^{(k)}_{\gamma_k} \in F^1 K^{(k)}.$$

Then the two representatives are related by

$$\delta^{(k)}_{p'_k}(\square'_{ij}) = g x g^{-1}.$$

Step 3: Conjugation is trivial modulo F^3 on F^2. Using the identity

$$g x g^{-1} = x[g, x],$$

we reduce the difference between gxg^{-1} and x to the commutator term $[g, x]$. Since

$$g \in F^1, \qquad x \in F^2,$$

Theorem B.3.2 yields

$$[g, x] \in [F^1, F^2] \subseteq F^3.$$

Hence

$$g x g^{-1} = x \cdot z \qquad \text{for some } z \in F^3.$$

Therefore

$$g x g^{-1} \equiv x \pmod{F^3}.$$

It follows that

$$\big[\delta^{(k)}_{p'_k}(\square'_{ij})\big] = \big[\delta^{(k)}_{p_k}(\square_{ij})\big] \quad \text{in} \quad F^2 K^{(k)} / F^3 K^{(k)}.$$

Equivalently,

$$\mathcal{A}^{(k)}_{ij}(p'_k) = \mathcal{A}^{(k)}_{ij}(p_k).$$

Step 4: Passage to the inverse limit. Since the equality above holds for every finite stage k, the two compatible families obtained from the two basepoint systems coincide in every component. Hence they define the same element of the inverse limit

$$K = \varprojlim_k \mathcal{K}_k.$$

Therefore the stabilized defect class

$$\kappa(u) \in K$$

is independent of the choice of admissible basepoint representatives. □

Theorem B.4.6 (Functorial uniqueness of the quadratic defect section)**.** *Let the hypotheses of Theorem B.4.4 hold, and let*

$$\kappa : U \to K$$

be the canonical quadratic defect section just constructed. Suppose

$$\sigma : U \to K$$

is another map satisfying the following properties:

(i) *for every*

$$g \in G = \mathrm{Aut}(U, \mathcal{C}) \qquad \textit{and} \qquad u \in U,$$

one has

$$\sigma(g \cdot u) = g_* \sigma(u);$$

(ii) *for each*

$$u \in U,$$

the value $\sigma(u)$ depends only on the first visible quadratic transport defect class of u, i.e. only on the corresponding element of the stabilized carrier K;

(iii) *on primitive quadratic square witnesses, σ agrees with the finite-stage defect assignment used to define κ.*

Then

$$\sigma = \kappa.$$

That is, κ is the unique natural quadratic defect assignment compatible with transport, filtration, and primitive quadratic normalization.

Proof. We prove equality pointwise. Fix

$$u \in U.$$

By Theorem B.4.4, the element

$$\kappa(u) \in K$$

is represented by a compatible family of finite-stage quadratic square classes

$$\big(\mathcal{A}_{ij}^{(k)}(p_k(u))\big)_k.$$

By construction of the stabilized carrier, K is generated by the stabilized images of such primitive quadratic square classes. Now consider

$$\sigma(u) \in K.$$

Property (ii) says that $\sigma(u)$ depends only on the first visible quadratic transport defect class of u. But this class is exactly the stabilized class represented by the same compatible family used to define $\kappa(u)$. Thus $\sigma(u)$ cannot depend on any additional structure beyond the element of K already encoded by $\kappa(u)$. Property (iii) fixes the normalization of σ on the primitive quadratic generators: on every primitive quadratic square witness, σ agrees with the defining assignment for κ. Since the stabilized carrier K is generated by these primitive quadratic classes, and since σ and κ agree on those generators while depending only on the resulting stabilized quadratic class, it follows that

$$\sigma(u) = \kappa(u).$$

Because $u \in U$ was arbitrary, we conclude that

$$\sigma = \kappa.$$

This proves uniqueness. □

B.5 Construction of the invariant

We now construct the confinement invariant itself.

Definition B.5.1 (Quadratic defect section)**.** Let

$$\kappa : U \to K$$

be the canonical degree–2 defect section of Theorem B.4.4.

Definition B.5.2 (The confinement invariant)**.** Define

$$\mathfrak{s}_{\mathrm{fus}} := Q \circ \kappa.$$

Theorem B.5.3 (Existence)**.** *The map*

$$\mathfrak{s}_{\mathrm{fus}} : U \to \mathbb{R}$$

is a well-defined scalar-valued function on the confinement state space.

Proof. To prove that $\mathfrak{s}_{\mathrm{fus}}$ is well defined, it suffices to verify that both factors in the composite

$$\mathfrak{s}_{\mathrm{fus}} = Q \circ \kappa$$

are well-defined maps with compatible source and target. First, by Theorem B.4.4, the quadratic defect section

$$\kappa : U \to K$$

is a canonically defined map on the confinement comparison world. The word "canonical" is essential: it means that $\kappa(u)$ depends only on the intrinsic transport data associated with the state u, and not on any auxiliary presentation, lift, or coordinate choice. Thus the expression $\kappa(u)$ makes sense for every $u \in U$. Second, by Theorem B.4.3, the scalar channel

$$Q : K \to \mathbb{R}$$

is a well-defined real-valued map on the stabilized quadratic carrier. Again, this means that once a degree–2 transport defect class has been formed as an element of K, its scalar image under Q is unambiguously determined. Therefore, for each $u \in U$, one may first form

$$\kappa(u) \in K$$

and then apply Q to obtain

$$Q(\kappa(u)) \in \mathbb{R}.$$

This defines a map

$$\mathfrak{s}_{\text{fus}} : U \to \mathbb{R}, \qquad \mathfrak{s}_{\text{fus}}(u) := Q(\kappa(u)).$$

No further compatibility condition is required, because the codomain of κ is precisely the domain of Q. Hence $\mathfrak{s}_{\text{fus}}$ is a well-defined scalar-valued function on U. $\square$

Theorem B.5.4 (Invariance under intrinsic evolution)**.** *Under the standing principle Standing Principle 1, let*

$$\Phi_t \in \mathrm{Aut}(U, \mathcal{C})$$

be an intrinsic evolution of the confinement system. Then

$$\mathfrak{s}_{\text{fus}}(\Phi_t(u)) = \mathfrak{s}_{\text{fus}}(u) \qquad \textit{for all } u \in U,\ t.$$

Proof. We prove the statement by unfolding, in order, the definitions of intrinsic evolution, naturality of the defect section, and admissibility of the scalar channel. Fix t and $u \in U$. By hypothesis,

$$\Phi_t \in \mathrm{Aut}(U, \mathcal{C}).$$

Thus Φ_t is an automorphism of the closed comparison world. Equivalently, Φ_t preserves all admissible comparison predicates and therefore belongs to the intrinsic symmetry group

$$G := \mathrm{Aut}(U, \mathcal{C}).$$

This is consistent with Theorem 11.2.4, which asserts precisely that intrinsic admissible transport-closed dynamics acts through automorphisms of the comparison world. Now apply the functoriality of the defect section. By Theorem B.4.4, for every $g \in G$,

$$\kappa(g \cdot u) = g_* \kappa(u),$$

where g_* denotes the induced action on the stabilized quadratic carrier K. Specializing to $g = \Phi_t$, we obtain

$$\kappa(\Phi_t(u)) = \Phi_{t*} \kappa(u).$$

Next use the admissibility of the scalar channel Q. By definition of admissible scalarization, and by the uniqueness theory cited in Theorem B.4.3, the map Q is invariant under the intrinsic symmetry action on K. That is,

$$Q(g_* k) = Q(k) \qquad \text{for all } g \in G,\ k \in K.$$

Again specializing to $g = \Phi_t$, we obtain

$$Q(\Phi_{t*} k) = Q(k) \qquad \text{for all } k \in K.$$

We now compute directly:

$$\begin{aligned} \mathfrak{s}_{\text{fus}}(\Phi_t(u)) &= (Q \circ \kappa)(\Phi_t(u)) \\ &= Q(\kappa(\Phi_t(u))) \\ &= Q(\Phi_{t*}\kappa(u)) \\ &= Q(\kappa(u)) \\ &= \mathfrak{s}_{\text{fus}}(u). \end{aligned}$$

Since $u \in U$ and t were arbitrary, the equality holds for all states and all intrinsic evolution times. Therefore $\mathfrak{s}_{\text{fus}}$ is invariant under intrinsic evolution. □

Theorem B.5.5 (Uniqueness up to scale)**.** *The scalar invariant $\mathfrak{s}_{\text{fus}}$ is unique up to multiplication by a nonzero real scalar.*

Proof. Let

$$\mathfrak{s}_1, \mathfrak{s}_2 : U \to \mathbb{R}$$

be two scalar invariants obtained by the construction of the appendix. By definition of the construction, there exist nonzero admissible scalarizations

$$Q_1, Q_2 : K \to \mathbb{R}$$

such that

$$\mathfrak{s}_1 = Q_1 \circ \kappa, \qquad \mathfrak{s}_2 = Q_2 \circ \kappa.$$

By Theorem B.4.3, the admissible scalarization space on K is one-dimensional. Therefore there exists

$$c \in \mathbb{R}^{\times}$$

such that

$$Q_1 = c\, Q_2.$$

Substituting this relation into the definition of $\mathfrak{s}_1$, we obtain

$$\mathfrak{s}_1 = Q_1 \circ \kappa = (c\, Q_2) \circ \kappa = c\,(Q_2 \circ \kappa) = c\, \mathfrak{s}_2.$$

Thus $\mathfrak{s}_1$ and $\mathfrak{s}_2$ differ by multiplication by a nonzero real constant. This proves uniqueness of the confinement invariant up to nonzero normalization. □

Corollary B.5.6 (Main appendix theorem)**.** *The confinement comparison world carries a scalar-valued function*

$$\mathfrak{s}_{\text{fus}} : U \to \mathbb{R}$$

which is invariant under intrinsic evolution and unique up to multiplication by a nonzero scalar.

Proof. The corollary is obtained by assembling Theorems B.5.3 to B.5.5. By Theorem B.5.3, the function

$$\mathfrak{s}_{\text{fus}} = Q \circ \kappa$$

is well defined on the confinement state space U. By Theorem B.5.4, this function is preserved under every intrinsic evolution

$$\Phi_t \in \text{Aut}(U, \mathcal{C}).$$

By Theorem B.5.5, any other scalar invariant obtained from the same transport stack differs from $\mathfrak{s}_{\text{fus}}$ by multiplication by a nonzero real scalar. These three statements are exactly the claims asserted in the corollary. □

B.6 Perturbative alignment criterion

Theorem B.5.6 is structural and nonperturbative. We now record the additional criterion under which one may identify it with the perturbative adiabatic invariant in the small-ϵ regime.

Theorem B.6.1 (Conditional carrier identification of the perturbative invariant)**.** *Under the standing principle Standing Principle 1, assume that in the perturbative regime*

$$\epsilon := \rho/L \ll 1,$$

the perturbative adiabatic invariant is obtained by admissible scalarization of the same transport stack, with leading nontrivial term carried by the first visible transport defect. Then the perturbative adiabatic invariant and the closed-system invariant

$$\mathfrak{s}_{\text{fus}} = Q \circ \kappa$$

are the same scalar channel written in different coordinates. Equivalently, under that perturbative identification, Kruskal's adiabatic invariant series is the asymptotic expansion of the unique admissible scalar channel on the first visible transport carrier.

Proof. The proof rests on four assertions:

(1) the transport filtration records perturbative order;

(2) the first visible scalar carrier occurs in degree 2;

(3) the scalar channel on that carrier is unique;

(4) under the stated perturbative hypothesis, no second independent scalar channel enters the same transport stack.

We treat them in order and then combine them. **Step 1: The filtration records perturbative order.** The small parameter

$$\epsilon := \rho/L$$

measures the ratio of the microscopic gyroradius scale to the macroscopic variation scale of the magnetic geometry. In the perturbative treatment of magnetic confinement, transport quantities are expanded as formal series in ϵ, and each order measures the first scale at which exact closure of the leading transport law fails. The intrinsic transport filtration

$$F^1 \supset F^2 \supset F^3 \supset \cdots$$

is organized by exactly the same principle. An element lies in F^m precisely when its transport defect is invisible through order $m-1$, and passage to the quotient

$$F^m/F^{m+1}$$

isolates the first nonvanishing defect at order m. Thus the perturbative order-ϵ^m data are encoded by the graded piece

$$F^m/F^{m+1}.$$

Step 2: The first visible scalar carrier is quadratic. By the main transport theory and, in particular, by the interface results (Theorems 13.10.2 and 13.10.4), the first visible nontrivial transport obstruction is degree 2. Equivalently, the first graded piece capable of carrying nontrivial scalar information is

$$K \cong F^2/F^3.$$

The degree–1 quotient carries only abelianized transport content and does not detect the intrinsic 2-skeletal obstruction. Hence the first nontrivial scalar invariant extracted from transport must factor through K. **Step 3: Uniqueness of the scalar channel on the first visible carrier.** By Theorem B.4.3, the space of admissible scalarizations of K in the realized scalar-channel regime is one-dimensional. Therefore any admissible scalar observable whose leading contribution is carried by the first visible transport defect must be proportional to the distinguished scalar channel

$$Q : K \to \mathbb{R}.$$

By the perturbative-identification hypothesis, the perturbative adiabatic invariant is of exactly this type. **Step 4: Identification.** Under the same perturbative-identification hypothesis, higher-order corrections still scalarize the same transport stack rather than opening a new independent scalar channel. By Theorem 13.10.7, every admissible degree–2 scalar detector descends uniquely through the quadratic carrier. Thus once the leading scalar transport content has been fixed on K, there is no second independent scalar channel through which that same transport stack can be scalarized. Consequently, the perturbative invariant and the intrinsic invariant

$$\mathfrak{s}_{\text{fus}} = Q \circ \kappa$$

arise from the same scalar channel and differ only by the perturbative coordinate expansion used to express that channel in the regime $\epsilon \ll 1$. This is the asserted identification. ∎

Corollary B.6.2 (Conditional all-orders asymptotic agreement)**.** *Under the hypotheses of Theorem B.6.1, in the perturbative regime*

$$\epsilon \ll 1,$$

the invariant $\mathfrak{s}_{\text{fus}}$ agrees with Kruskal's adiabatic invariant series to all orders.

Proof. By Theorem B.6.1, under the stated perturbative identification the perturbative invariant series and the intrinsic invariant $\mathfrak{s}_{\text{fus}}$ are two presentations of the same scalar channel. Therefore their asymptotic coefficients agree order by order in the small-ϵ regime. ∎

Remark B.6.3*.* The point is not that $\mathfrak{s}_{\text{fus}}$ is defined perturbatively. It is not. Rather, once $\mathfrak{s}_{\text{fus}}$ has been defined intrinsically by the closed-system transport stack, any perturbative invariant satisfying Theorem B.6.1 is recovered as a local coordinate expansion of that same scalar channel in the regime $\epsilon \ll 1$.

B.7 Conditional confinement package

The theorem-level output of the appendix is the intrinsic scalar $\mathfrak{s}_{\text{fus}}$. We now record the perturbative and confinement readings that become available once the relevant interpretation hypotheses are fixed.

Theorem B.7.1 (Conditional confinement interpretation package)**.** *Under the standing principle Standing Principle 1, let $(U, \mathcal{C})$ be a magnetic geometry realized as a closed comparison world. Then the following statements hold.*

(i) *There exists a scalar-valued function*

$$\mathfrak{s}_{\text{fus}} : U \to \mathbb{R}$$

which is invariant under intrinsic evolution and unique up to multiplication by a nonzero real scalar.

(ii) *Under the additional perturbative-identification hypothesis of Theorem B.6.1, the perturbative adiabatic invariant is the asymptotic expansion of* $\mathfrak{s}_{\mathrm{fus}}$ *in the regime* $\epsilon \ll 1$.

(iii) *If confinement for the relevant particle family is modeled by compact containment of the appropriate invariant level sets, then the resulting structural confinement criterion is that the magnetic geometry confines exactly when those level sets of* $\mathfrak{s}_{\mathrm{fus}}$ *are compact and remain contained in the plasma volume.*

(iv) *Under the same confinement model, the corresponding optimization problem is to optimize the confinement quality of the level sets of* $\mathfrak{s}_{\mathrm{fus}}$ *over the admissible comparison structures induced by the geometry.*

Proof. Part (i) is exactly Theorem B.5.6. Part (ii) is exactly Theorem B.6.1. For parts (iii) and (iv), once confinement is modeled by compact containment of invariant level sets, the statements are the formal restatement of that model using the theorem-level scalar $\mathfrak{s}_{\mathrm{fus}}$ singled out by part (i). Because changing magnetic geometry changes the induced comparison world, and hence changes the resulting invariant $\mathfrak{s}_{\mathrm{fus}}$, optimizing confinement inside that model is equivalently the problem of optimizing the geometry of the relevant $\mathfrak{s}_{\mathrm{fus}}$-level sets across the induced comparison structures. □

Remark B.7.2. The theorem does not yet provide a reactor-specific closed-form number for $\mathfrak{s}_{\mathrm{fus}}$ for an arbitrary coil set. What it provides is the canonical intrinsic scalar to which any later perturbative, numerical, or confinement model must refer.

B.8 Magnetic-confinement interpretation

Theorem B.7.1 separates theorem-level invariant content from the later confinement reading. Under the confinement interpretation package, the structure may be read as follows.

(i) The gyration of a charged particle in a magnetic field is an internal transport process of the confinement world.

(ii) The failure of this transport to close exactly is a loop defect. Its first stable visible component lies in the quadratic carrier

$$F^2/F^3.$$

(iii) The map

$$\kappa : U \to K$$

assigns to each state its intrinsic quadratic transport defect class.

(iv) The scalar channel

$$Q : K \to \mathbb{R}$$

extracts the unique admissible scalar carried by that defect in the realized scalar-channel regime used in the appendix.

(v) The composite

$$\mathfrak{s}_{\text{fus}} = Q \circ \kappa$$

is therefore the canonical intrinsic transport scalar supplied by the closed-system structure for this application.

Thus the intrinsic transport scalar is not postulated at the carrier level. Its later reading as a nonperturbative adiabatic invariant or confinement observable is a downstream interpretation of the scalar channel of the quadratic transport carrier.

Remark B.8.1. No Liouville-integrability assumption enters. No continuous-symmetry assumption enters. No data-driven learning step enters. All three are replaced at the carrier level by the closed-system transport stack already proved in the main text.

B.9 Consequences of the interpretation package

Once the intrinsic scalar $\mathfrak{s}_{\text{fus}}$ is established theorem-level and the perturbative/level-set interpretation package is adopted, several immediate consequences follow.

B.9.1 General-geometry existence

The existence of $\mathfrak{s}_{\text{fus}}$ is not restricted to integrable or symmetric magnetic fields. Its construction depends only on the closed comparison structure and the quadratic scalar channel. Accordingly, any magnetic geometry that can be realized as a closed comparison world carries the same canonical nonperturbative transport scalar.

B.9.2 Structural confinement criterion under level-set interpretation

Because $\mathfrak{s}_{\text{fus}}$ is invariant under intrinsic evolution, if confinement is modeled by compact containment of the relevant invariant level sets, then confinement may be rephrased geometrically in terms of the level sets of $\mathfrak{s}_{\text{fus}}$. The resulting structural criterion is:

> A magnetic geometry confines the relevant particles exactly when the appropriate level sets of $\mathfrak{s}_{\text{fus}}$ are compact and remain contained within the plasma volume.

This criterion is downstream: it depends on the theorem-level scalar channel of the transport filtration together with the chosen level-set interpretation, not on full-orbit simulation.

B.9.3 Optimization reformulation under the same criterion

Changing magnetic geometry changes the comparison predicates. Changing the comparison predicates changes the transport filtration. Changing the transport filtration changes the quadratic carrier and therefore changes the scalar channel $\mathfrak{s}_{\mathrm{fus}}$. Thus, under the same level-set confinement model, the optimization problem becomes:

> Find the magnetic geometry whose induced comparison structure optimizes the compactness and confinement properties of the level sets of $\mathfrak{s}_{\mathrm{fus}}$.

This reformulates the task as a variational problem on the space of comparison structures rather than only as a brute-force orbit-search problem.

B.9.4 Further confinement regimes

Nothing in the theorem-level construction of $\mathfrak{s}_{\mathrm{fus}}$ used a special property of alpha particles beyond the existence of a closed comparison structure for the confinement system. Accordingly, the same structural machinery applies, in the same conditional sense, to any confinement problem whose relevant state space is describable as a closed comparison world with transport filtration and quadratic scalar channel.

B.10 Normalized exact observables

The scalar channel Q is unique only up to multiplication by a nonzero constant. Accordingly, absolute values of $\mathfrak{s}_{\mathrm{fus}} = Q \circ \kappa$ depend on the chosen normalization. What is canonically determined without any further choice is the ratio of two nonzero scalar values.

Proposition B.10.1 (Normalized exact invariant)**.** *Let*

$$Q : K \to \mathbb{R}_{\geq 0}$$

be the canonical scalar channel on the stabilized quadratic carrier, and let

$$u_* \in U$$

satisfy

$$\kappa(u_*) \neq 0.$$

Then the normalized scalar

$$\widehat{\mathfrak{s}}_{\mathrm{fus}}(u; u_*) := \frac{Q(\kappa(u))}{Q(\kappa(u_*))}$$

is well defined, independent of the normalization of Q, and invariant under intrinsic evolution.

Proof. There are three claims to establish:

(i) the ratio is independent of normalization of Q;

(ii) the ratio is defined, i.e. the denominator is nonzero;

(iii) the ratio is invariant under intrinsic evolution.

Step 1: Independence of normalization. Let

$$Q' = cQ$$

for some $c > 0$. Then for every $u \in U$,

$$Q'(\kappa(u)) = c\,Q(\kappa(u)),$$

and similarly

$$Q'(\kappa(u_*)) = c\,Q(\kappa(u_*)).$$

Therefore

$$\frac{Q'(\kappa(u))}{Q'(\kappa(u_*))} = \frac{cQ(\kappa(u))}{cQ(\kappa(u_*))} = \frac{Q(\kappa(u))}{Q(\kappa(u_*))}.$$

Thus the normalized scalar does not depend on the chosen nonzero normalization of the scalar channel. **Step 2: Nonvanishing of the denominator.** By hypothesis,

$$\kappa(u_*) \neq 0.$$

The scalar channel Q is by construction nontrivial on the realized carrier and takes nonnegative values on the chosen normalization. The intended normalization in this proposition is one for which nonzero carrier classes are assigned nonzero nonnegative values. Under that normalization,

$$Q(\kappa(u_*)) > 0.$$

Hence the denominator is nonzero and the ratio is well defined. **Step 3: Invariance under intrinsic evolution.** Let

$$\Phi_t \in \mathrm{Aut}(U, \mathcal{C}).$$

By Theorem B.5.4,

$$Q(\kappa(\Phi_t(u))) = Q(\kappa(u))$$

for every $u \in U$. Applying this once to u and once to u_*, we obtain

$$Q(\kappa(\Phi_t(u))) = Q(\kappa(u)), \qquad Q(\kappa(\Phi_t(u_*))) = Q(\kappa(u_*)).$$

Therefore

$$\widehat{\mathfrak{s}}_{\mathrm{fus}}(\Phi_t(u); \Phi_t(u_*)) = \frac{Q(\kappa(\Phi_t(u)))}{Q(\kappa(\Phi_t(u_*)))} = \frac{Q(\kappa(u))}{Q(\kappa(u_*))} = \widehat{\mathfrak{s}}_{\mathrm{fus}}(u; u_*).$$

This proves invariance of the normalized scalar. □

Remark B.10.2. The proposition identifies the first exactly computable scalar quantity determined entirely by the theorem stack without fixing an external normalization convention.

B.11 Structural summary

The argument of the appendix is:

$$
\begin{aligned}
\text{closed confinement system} &\Longrightarrow \text{comparison completeness} \\
&\Longrightarrow \text{canonical quotient semantics} \\
&\Longrightarrow \text{transport filtration } F^1 \supset F^2 \supset F^3 \supset \cdots \\
&\Longrightarrow \text{stabilized quadratic carrier } K \\
&\Longrightarrow \text{unique scalar channel } Q \\
&\Longrightarrow \text{intrinsic scalar } \mathfrak{s}_{\text{fus}}.
\end{aligned}
$$

Through the intrinsic scalar $\mathfrak{s}_{\text{fus}} = Q \circ \kappa$, every arrow in this chain is theorem-level within the closed relational stack, using the inherited realized scalar-channel package already isolated upstream. Thus $\mathfrak{s}_{\text{fus}}$ is not postulated, not learned from data, and not restricted to integrable geometries at the level of its intrinsic construction. Perturbative matching and confinement-level interpretation are additional downstream reading layers built on that scalar.

B.12 Conclusion

This appendix has one theorem-level purpose: to exhibit the downstream confinement use of the quadratic scalar channel of the closed-system transport theory. The chain proved here is:

$$
\begin{aligned}
(U, \mathcal{C}) &\Longrightarrow N = F^1 \\
&\Longrightarrow V = F^1/F^2 \\
&\Longrightarrow \Lambda^2(V) \xrightarrow{q_\wedge} F^2/F^3 \\
&\Longrightarrow K = \mathcal{K}_\infty \\
&\Longrightarrow Q : K \to \mathbb{R} \\
&\Longrightarrow \mathfrak{s}_{\text{fus}} = Q \circ \kappa.
\end{aligned}
$$

Thus the abstract machinery of the main text yields a canonical intrinsic transport scalar in the magnetic-confinement setting whenever the system is realized as a closed comparison world. The point is not merely that the scalar can be named. The point is that its existence and uniqueness up to scale are fixed by the internal structure of that world. Perturbative matching to Kruskal-type series and any confinement-level or optimization reading remain separate downstream interpretation steps. In this sense, the appendix closes the manuscript's forward arc: the abstract chain from closure, transport, and quadratic obstruction to scalar channel is carried into a downstream confinement application template.

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