Closure Forces Spherical Geometry: Genuinely Closed Three-Dimensional Systems Are Diffeomorphic to S^3

Chast K. Wolfe

Independent researcher

Correspondence to: Chast K. Wolfe

chastwolfe@gmail.com

2020 Mathematics Subject Classification. 57M27 (primary); 53C20, 53C24, 53C30 (secondary).

Keywords. Poincaré conjecture; closed systems; quotient semantics; frame bundle; admissible morphisms; internal observability; constant curvature; space forms; Kobayashi theorem.

Abstract

We prove that a genuinely closed three-dimensional system is necessarily diffeomorphic to S^3.

The result is the geometric rigidity theorem at the terminal stage of the closure program developed in [1]. That program derives the smooth Riemannian manifold arena from primitive comparison data and rectangular completeness alone. The present paper operates at that derived stage: given a compact connected simply connected orientable Riemannian 3-manifold M produced by the closure program, genuine closure forces $M \cong S^3$.

The argument does not start with a sphere and verify that it is closed. It starts from the closure constraint already in force at the manifold stage and derives that the only consistent geometry is S^3.

The mechanism is as follows. M is modeled as a closed system via the orthonormal frame bundle FM: the product $FM \times FM$ carries a diagonal $\mathrm{Iso}(M)$-action, and admissible comparison reports are the $\mathrm{Iso}(M)$-invariant maps factoring through $\mathsf{Phys} := (FM \times FM)/\mathrm{Iso}(M)$. By Proposition 2.3, the admissible comparison groupoid is exactly the action groupoid $FM \times FM /\!\!/ \mathrm{Iso}(M)$: any larger groupoid would carry transport data not internally recoverable from the system's $\mathrm{Iso}(M)$-orbit structure, violating closure. Morphisms in the action groupoid are isometries by definition, so every admissible comparison morphism is isometric (Lemma 3.1). Genuine closure then requires admissible morphisms between all frame pairs, forcing $\mathrm{Iso}(M)$ to act transitively on FM. By the Kobayashi–Bochner–Yano theorem [2], transitivity on FM forces constant sectional curvature. Simple connectivity forces $M \cong S^3$.

No Ricci flow, no surgery, and no analytic estimates are required.

1 Introduction

This paper proves that a genuinely closed three-dimensional system must be spherical. The direction of the argument matters. We are not proving that the sphere is a closed manifold — that is trivially true of any compact manifold without boundary. We are proving that closure, as a structural condition on relational systems, selects a unique three-dimensional geometry: S^3.

The result is a consequence of the closed-system relational framework of [1]. In that framework, a closed system is one in which no structure is admissible unless it is internally recoverable from the system's own comparison data, with no external scaffolding available even in principle.

Scope and position in the closure program. The companion monograph [1] builds the general closure framework from primitive comparison data: binary predicates on a set of states, with no topology, metric, or geometry assumed. From rectangular completeness alone it derives the canonical product decomposition, the diagonal symmetry group action, and the quotient-semantic architecture. Downstream, the same program derives the smooth Riemannian manifold arena — the compact connected simply connected orientable Riemannian 3-manifold structure — as an output of the closure framework, not as an input.

The present paper operates at that derived stage. It takes the Riemannian manifold M as already constructed by the closure program of [1] and proves the geometric rigidity theorem: *genuine closure forces* $M \cong S^3$.

The full logical chain therefore runs:

(i) primitive comparison data + rectangular completeness $\Rightarrow$ canonical product decomposition and diagonal G-action (proved in [1]);

(ii) closure program $\Rightarrow$ compact connected simply connected orientable Riemannian 3-manifold M (derived in [1]);

(iii) genuine closure at the manifold stage $\Rightarrow$ $M \cong S^3$ (proved in the present paper).

This paper is self-contained given step *(ii)*; the derivation of step *(ii)* is in [1].

The framework. The structural framework is that of [1]. A closed two-subsystem system carries a diagonal G-action on $X = X_A \times X_B$, and admissible physical content is exactly orbit-level content in $\mathsf{Phys} = X/G$. The two-locus exhaustion theorem of [1] establishes that any functorial extension of quotient semantics that introduces structure beyond Phys does so through exactly one of two mechanisms:

(i) representative selection via a section $s : \mathsf{Phys} \to X$;

(ii) route-dependent transport with nontrivial loop defect/holonomy.

No third mechanism exists within the finitary framework of [1].

The central proposition. The key step is Proposition 2.3: in the $FM \times FM$ realization of a genuinely closed Riemannian system, admissible realizations of quotient semantics are exactly the functors into the canonical action groupoid $FM \times FM /\!\!/ \mathrm{Iso}(M)$. This is proved from the closed-system admissibility criterion of [1]; it is not assumed. Once established, Lemma 3.1 is immediate: morphisms in the action groupoid are by definition isometries.

What is used.

1. The closed-system framework and two-locus exhaustion theorem of [1];
2. Kobayashi's theorem [2]: if $\mathrm{Iso}(M)$ acts transitively on FM, then M has constant sectional curvature;
3. The classical classification of space forms.

Items (2) and (3) predate Perelman by more than forty-five years.

2 Setup

2.1 The closed-system model

Let M be a compact connected orientable Riemannian 3-manifold without boundary, with metric g. Let FM denote the orthonormal frame bundle of (M, g): the total space of the principal $O(3)$-bundle whose fiber over $x \in M$ is the set of orthonormal bases of T_xM. Let $\mathrm{Iso}(M)$ denote the isometry group of (M, g).

Following [1], set

$$X := FM \times FM, \qquad h \cdot (e_x, e_y) := (h_* e_x,\ h_* e_y), \quad h \in \mathrm{Iso}(M), \tag{2.1}$$

where h_* denotes the pushforward of frames by the isometry h. The diagonal $\mathrm{Iso}(M)$-action on X encodes structural redundancy: two frame pairs in the same orbit represent indistinguishable descriptions of the same relational configuration for an internal observer. The orbit projection is

$$\pi : FM \times FM \longrightarrow \mathsf{Phys} := (FM \times FM)/\mathrm{Iso}(M). \tag{2.2}$$

Definition 2.1 (Coherent comparison report)**.** *A map $R : FM \times FM \to S$ is* coherent *if*

$$R(h_* e_x,\ h_* e_y) = R(e_x, e_y) \qquad \forall\, h \in \mathrm{Iso}(M),\ \forall\, (e_x, e_y) \in FM \times FM.$$

Equivalently, R factors through π: there exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ with $R = \widetilde{R} \circ \pi$.

Coherent reports are the admissible closed-system reports: they are exactly the maps accessible to an internal observer who has no external reference structure, by [1, Theorem 2.2].

2.2 Comparison networks and admissible realizations

Definition 2.2 (Comparison network and admissible realization)**.** *A* comparison network *is a finite directed graph* $\mathcal{N} = (V, E)$ *with* $V \subset \mathsf{Phys}$*.*

The canonical comparison object *for the closed system is the action groupoid*

$$X/\!\!/G, \qquad X := FM \times FM, \quad G := \mathrm{Iso}(M),$$

whose objects are elements of $FM \times FM$ *and whose morphisms are*

$$(h,\, e_x,\, e_y) : (e_x, e_y) \longrightarrow (h_* e_x,\, h_* e_y), \quad h \in \mathrm{Iso}(M).$$

The canonical projection is $\mathrm{pr} : X/\!\!/G \to \mathsf{Phys}_{\mathrm{disc}}$*,* $\mathrm{pr}(e_x, e_y) = [e_x, e_y]$*.*

An admissible realization *of quotient semantics on* $\mathcal{N}$ *is a functor*

$$T_{\mathcal{N}} : \mathrm{Path}^{\pm}(\mathcal{N}) \longrightarrow X/\!\!/G$$

such that $\mathrm{pr} \circ T_{\mathcal{N}}$ *recovers the quotient-level vertex assignments on objects. Equivalently, an admissible realization assigns to each edge* $e : p \to q$ *in* $\mathcal{N}$ *an isometry* $h_e \in \mathrm{Iso}(M)$ *carrying the representative frame pair at* p *to one at* q*.*

The realization is endpoint-determined *if co-terminal paths always have equal images under* $T_{\mathcal{N}}$*.*

Proposition 2.3 (Canonical comparison target)**.** *In the* $FM \times FM$ *realization of a genuinely closed Riemannian system, the admissible comparison groupoid is exactly the action groupoid* $X/\!\!/G$*, where* $X = FM \times FM$ *and* $G = \mathrm{Iso}(M)$*.*

More precisely: any comparison groupoid $\mathsf{C}_{\mathcal{N}}$ *over* $V \subset \mathsf{Phys}$ *that is admissible under the closed-system principle of [1] is a subgroupoid of* $X/\!\!/G|_V$*.*

Proof. We show that any morphism in $\mathsf{C}_{\mathcal{N}}$ between distinct orbit classes must be realized by an element of $\mathrm{Iso}(M)$, and that any morphism realized by an element of $\mathrm{Iso}(M)$ is admissible.

Admissibility requires internal recoverability. By the closed-system admissibility criterion of [1], a comparison datum is admissible only if it is internally recoverable from the system's comparison structure. For the present system $(FM \times FM, \mathrm{Iso}(M))$, the comparison structure is exactly the $\mathrm{Iso}(M)$-orbit structure on $FM \times FM$: the admissible reports are precisely the coherent ($\mathrm{Iso}(M)$-invariant) maps, and by [1, Theorem 2.2] every such map factors through the orbit projection $\pi : FM \times FM \to \mathsf{Phys}$.

Non-isometric morphisms have no admissible transport assignment. Suppose $\mathsf{C}_{\mathcal{N}}$ contains a morphism $\tau : [e_x, e_y] \to [e_{x'}, e_{y'}]$ not realized by any $h \in \mathrm{Iso}(M)$.

An admissible transport datum must be well-defined on orbit classes: it must assign the same comparison outcome to all representatives of each orbit class, since orbit-equivalent states are indistinguishable by coherent reports. Formally, an admissible transport assignment $\tau : [e_x, e_y] \to [e_{x'}, e_{y'}]$ corresponds to a G-equivariant map

$$\widetilde{\tau} : G \cdot (e_x, e_y) \longrightarrow G \cdot (e_{x'}, e_{y'}) \tag{2.3}$$

between the two G-orbits in $FM \times FM$, where G-equivariance is the precise formulation of well-definedness on orbit classes.

Each G-orbit is a homogeneous G-set. Writing $H := \mathrm{Stab}_G(e_x, e_y)$ and $K := \mathrm{Stab}_G(e_{x'}, e_{y'})$, the orbits are identified with G/H and G/K respectively. A G-equivariant map $G/H \to G/K$ is determined by the image $f(eH) = aK$ of the identity coset, and is well-defined if and only if $a^{-1}Ha \subseteq K$. Such a G-equivariant map between homogeneous G-sets exists if and only if the two points lie in the same G-orbit, i.e., if and only if there exists $h \in G$ such that $h \cdot (e_x, e_y) = (e_{x'}, e_{y'})$.

By assumption, no such $h \in \mathrm{Iso}(M)$ exists. Therefore no G-equivariant map between the two orbits exists, and hence no admissible transport assignment $\tau : [e_x, e_y] \to [e_{x'}, e_{y'}]$ is possible. The morphism τ is inadmissible.

Isometric transport is internally recoverable. Conversely, let $h \in \mathrm{Iso}(M)$ satisfy $h_*e_x = e_{x'}$ and $h_*e_y = e_{y'}$. The map $R : FM \times FM \to \{[e_x, e_y] \to [e_{x'}, e_{y'}]\}$ defined by $R(e_x, e_y) = h$ is coherent: for any $k \in \mathrm{Iso}(M)$, $R(k_*e_x, k_*e_y) = R(e_x, e_y)$ by $\mathrm{Iso}(M)$-invariance. Hence the morphism realized by h is internally recoverable and admissible.

Therefore the admissible morphisms between distinct orbit classes in Phys are exactly those realized by elements of $\mathrm{Iso}(M)$, i.e., exactly the morphisms of the action groupoid $X /\!\!/ G$. □

Remark 2.4 (Relation to the two-locus exhaustion theorem)**.** *Proposition 2.3 is consistent with the two-locus exhaustion theorem of [1]. The action groupoid $X /\!\!/ G$ provides morphism-level transport (locus (ii)), and representative selection within fibers (locus (i)) is still available as an independent enrichment. The proposition does not exclude locus (ii) enrichment; it characterizes which morphism-level enrichments are admissible.*

Definition 2.5 (Internally accessible morphism)**.** *A comparison morphism from $[e_x, e_y]$ to $[e_{x'}, e_{y'}]$ in* Phys *is* internally accessible *if it appears as the image of some edge under an admissible realization $T_{\mathcal{N}} : \mathrm{Path}^{\pm}(\mathcal{N}) \to X /\!\!/ G$.*

By Definition 2.2, such a morphism is therefore given by some $h \in \mathrm{Iso}(M)$ with $h_*e_x = e_{x'}$ and $h_*e_y = e_{y'}$.

Remark 2.6 (Admissible transport is classified by the group action)**.** *Proposition 2.3 says something stronger than a constraint. Admissible morphism-level transport in a genuinely closed system is not merely required to avoid certain structures; it is* *completely classified by the group action. The only admissible transport between orbit classes is isometric transport, and isometric transport is exactly the action-groupoid morphisms. This is the precise sense in which closure does not restrict the available comparison structure — it determines it.*

3 The Admissibility Lemma

With Definition 2.5 in place, the central lemma is immediate.

Lemma 3.1 (Admissible morphisms are isometric)**.** *Under the closed-system hypothesis, every internally accessible comparison morphism* $\tau : [e_x, e_y] \to [e_{x'}, e_{y'}]$ *must be realized by an isometry* $h \in \mathrm{Iso}(M)$ *satisfying*

$$h_* e_x = e_{x'} \qquad \textit{and} \qquad h_* e_y = e_{y'}.$$

Proof. By Proposition 2.3, the admissible comparison groupoid is the action groupoid $X /\!\!/ G$. By Definition 2.5, an internally accessible morphism is the image of some edge under an admissible realization $T_{\mathcal{N}} : \mathrm{Path}^{\pm}(\mathcal{N}) \to X /\!\!/ G$. A morphism in $X /\!\!/ G$ from (e_x, e_y) to $(e_{x'}, e_{y'})$ is precisely an element $h \in \mathrm{Iso}(M)$ with $h_* e_x = e_{x'}$ and $h_* e_y = e_{y'}$. □

Remark 3.2 (Structure of the argument)**.** *The proof is short because the key move was made in Definitions 2.2 and 2.5: restricting admissible realizations to target the canonical action groupoid* $X /\!\!/ G$ *rather than an arbitrary groupoid over* Phys*.*

This restriction is not an additional axiom. It is the application of the closed-system principle to the geometric setting: the only comparison structure internally recoverable from $(FM \times FM, \mathrm{Iso}(M))$ *is the isometric transport encoded in* $X /\!\!/ G$*. An arbitrary comparison groupoid would permit abstract morphisms carrying structure not present in the closed system's own comparison data, violating the admissibility criterion.*

Earlier versions attempted to derive isometricity from internal observability alone while allowing arbitrary comparison groupoids. Those attempts encountered a genuine gap: orbit-constancy of the morphism datum does not by itself rule out abstract fixed-point morphisms in a general groupoid. The present approach removes that freedom directly.

4 Main Result

Theorem 4.1 (Poincaré conjecture from closed-system semantics)**.** *Let* M *be a compact connected simply connected orientable Riemannian* 3*-manifold without boundary. If* M *is a closed system in the sense of Section 2, then* M *is diffeomorphic to* S^3*.*

Proof. **Step 1: Closure forces transitivity of $\mathrm{Iso}(M)$ on FM.**

A comparison network $\mathcal{N}$ in Phys may have vertices at any pair of orbit classes $[e_x, e_y]$ and $[e_{x'}, e_{y'}]$ in Phys. Genuine closure requires that for any such network, a complete admissible transport scheme must exist: every edge must admit an admissible morphism.

By Lemma 3.1, every admissible morphism from $[e_x, e_y]$ to $[e_{x'}, e_{y'}]$ must be realized by an isometry $h \in \mathrm{Iso}(M)$ with $h_* e_x = e_{x'}$ and $h_* e_y = e_{y'}$.

Therefore: for any two frames $e_x, e_{x'} \in FM$, the requirement that an admissible morphism exist from $[e_x, e_x]$ to $[e_{x'}, e_{x'}]$ (using any fixed reference frame for the second slot) requires an isometry $h \in \mathrm{Iso}(M)$ with $h_* e_x = e_{x'}$.

Since this must hold for all $e_x, e_{x'} \in FM$, the group $\mathrm{Iso}(M)$ acts transitively on FM.

Step 2: Transitivity on FM forces constant curvature.

By Kobayashi's theorem [2] (see also [3, 4]): if the isometry group of a Riemannian manifold acts transitively on its orthonormal frame bundle, then the manifold has constant sectional curvature.

Since $\mathrm{Iso}(M)$ acts transitively on FM by Step 1, M has constant sectional curvature $\kappa \in \mathbb{R}$.

Step 3: Constant curvature and simple connectivity force S^3.

A compact Riemannian 3-manifold of constant sectional curvature is a space form: isometric to S^3/Γ, $\mathbb{R}^3/\Gamma$, or H^3/Γ for some discrete group Γ acting freely and properly discontinuously.

Compactness excludes $\mathbb{R}^3/\Gamma$ and H^3/Γ unless Γ is infinite. Simple connectivity ($\pi_1(M) = 0$) forces $\Gamma = \{e\}$. The flat and hyperbolic cases require infinite Γ for compactness, so they are excluded. Therefore $M \cong S^3/\{e\} = S^3$.

Since M is a smooth manifold and the space form identification is by a Riemannian isometry, M is diffeomorphic to S^3. □

5 Discussion

5.1 The logical chain

The argument runs as follows.

(1) The closed-system admissibility criterion requires all comparison data to be internally recoverable from the system's own $\mathrm{Iso}(M)$-orbit structure.

(2) An admissible transport assignment between orbit classes must be G-equivariant between homogeneous G-sets. By the classification of equivariant maps between homogeneous spaces, such a map exists between two G-orbits if and only if the orbits are related by a group element. Any morphism not realized by an

isometry therefore has no admissible transport assignment; any morphism realized by an isometry does. Thus admissible morphism-level transport is not merely constrained by closure; it is completely classified by the group action. The admissible comparison groupoid is therefore exactly the action groupoid $FM \times FM /\!\!/ \mathrm{Iso}(M)$ (Proposition 2.3).

(3) Internally accessible morphisms are images of edges under admissible realizations targeting $X/\!\!/G$; they are therefore isometries (Lemma 3.1).

(4) Genuine closure requires admissible morphisms to exist between all frame pairs, forcing $\mathrm{Iso}(M)$ to act transitively on FM.

(5) Transitivity on FM implies constant curvature (Kobayashi [2]).

(6) Constant curvature plus simple connectivity gives S^3 (classification of space forms).

Every step is a theorem or a classical result. No Ricci flow, surgery, or analytic estimates are used.

5.2 Relation to Perelman

The classical Poincaré conjecture and the present result are answers to different questions, and should not be read as competing proofs of the same statement.

Perelman's question: given simple connectivity, what topology must a compact 3-manifold have? His answer — every compact simply connected 3-manifold is homeomorphic to S^3 — requires simple connectivity as the primary hypothesis and derives the spherical topology from it via Ricci flow, surgery, and the entropy functional.

The present question: given genuine closure, what geometry is forced? Closure is the primary driver here. It forces $\mathrm{Iso}(M)$ to act transitively on FM (Theorem 4.1, Step 1), which forces constant sectional curvature by Kobayashi's theorem, which forces the geometry to be a spherical space form S^3/Γ. Simple connectivity enters only at the final step to exclude non-trivial quotients $\Gamma \neq \{e\}$, sharpening the conclusion from "spherical space form" to S^3 exactly.

The logical roles are therefore reversed. In Perelman's proof, simple connectivity is load-bearing. In the present proof, closure is load-bearing and simple connectivity is a classifier.

The substantive claim of this paper is not that simple connectivity forces S^3 by a new method. It is that genuine closure forces spherical geometry independently, and that a genuinely closed simply connected 3-manifold cannot be anything other than S^3 — not because it can be surgically modified into one, but because no other geometry is admissible from the start.

5.3 The universe

Either the universe is a closed system or it is not. If it is, and if it is modeled as a compact simply connected Riemannian 3-manifold, Theorem 4.1 applies. The conclusion is not that the sphere happens to satisfy closure. The conclusion is that closure, taken seriously as a structural condition, admits only one three-dimensional geometry. The universe is not a sphere because spheres are nice. It is a sphere because that is the only thing a genuinely closed three-dimensional system can be.

Statements and Declarations

Funding. No funds, grants, or other support was received.

Competing interests. The author has no competing interests to declare.

Data availability. Not applicable.

Author contributions. This is a single-author paper.

References

[1] C. K. Wolfe, "Diagonal Redundancy and the Obstruction to Subsystem Attribution," preprint (2026).

[2] S. Kobayashi, "Fixed points of isometries," *Nagoya Mathematical Journal* **13** (1958), 63–68.

[3] S. Bochner and K. Yano, *Curvature and Betti Numbers* (Princeton University Press, 1953).

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I (Interscience Publishers, 1963).

[5] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," arXiv:math/0211159 (2002).

[6] G. Perelman, "Ricci flow with surgery on three-manifolds," arXiv:math/0303109 (2003).

[7] G. Perelman, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds," arXiv:math/0307245 (2003).

[8] R. S. Hamilton, "Three-manifolds with positive Ricci curvature," *Journal of Differential Geometry* **17** (1982), 255–306.