Closed Comparison Worlds and the Obstruction to Subsystem Attribution

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Abstract. This paper gives a finitary structural account of subsystem attribution in closed two-subsystem comparison worlds. The primitive data are a set of joint states equipped only with binary comparison predicates; no topology, metric, dynamics, background geometry, or gauge structure is assumed. These data determine left- and right-profile spaces, the intrinsic automorphism group, and the induced diagonal redundancy. *Rectangular completeness* is exactly the condition that the canonical profile map onto the product of the two profile spaces is bijective; it is the minimal internal condition forcing the canonical product decomposition and, among locally distinguishable worlds, is equivalent to profile-maximality against proper profile-determined one-point extensions. The resulting diagonal action yields an orbit quotient $\mathsf{Phys} = X/G$. Under the closed-world admissibility criterion, admissible state reports are precisely the maps that factor through this quotient. Hence pure diagonal-orbit motion has no invariant subsystem attribution: quotient-level reports cannot decide whether such a change is assigned to subsystem A or to subsystem B. Finally, for finite comparison networks, functorial extensions of quotient semantics admit only two enrichment loci. Non-endpoint-determined behavior occurs if and only if the morphism-level transport data have nontrivial loop defect, equivalently nontrivial holonomy; any object-level supplement is representative selection within orbits. Consequently, within the stated finitary extension formalism, there is no third independent enrichment locus.

Keywords: diagonal redundancy, comparison worlds, rectangular completeness, local distinguishability, profile-maximality, orbit quotient, quotient semantics, subsystem attribution, pure orbit motion, representative selection, route-dependent transport, holonomy, loop defect, functorial extensions of quotient semantics

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1 Closed Comparison Worlds with Diagonal Redundancy

Can one say, in a closed composite system, which subsystem has moved when no external frame is available? Under the closure and admissibility hypotheses below, pure orbit motion admits no invariant subsystem attribution. The paper also identifies the additional places where structure can enter beyond that conclusion and proves that the list is exhaustive within the stated finitary framework.

What a two-subsystem world is. A two-subsystem comparison world is the formal abstraction of a pairwise description whose available content consists of comparisons between joint states of a system A and a system B: an electron and a detector, two gravitating bodies, two charged particles, or any paired subsystems treated relationally. Pairwise comparison is a basic relational unit in many force-law, measurement, and scattering descriptions. No particular force, dynamics, metric, topology, or background frame is built into the setup. The question is what follows for such a pairwise description once admissible structure is restricted to what is recoverable from the comparison data themselves. The formal results below therefore apply to pairwise descriptions satisfying the stated structural hypotheses.

What closure means here. The result turns on the meaning of *closed*. Here closure is a criterion of admissibility: a structure is admissible only if it is internally recoverable from the system's own comparison data, with no external scaffolding even in principle. For a two-subsystem universe, this criterion has a minimal formal expression. The minimal internal condition forcing a canonical product decomposition $U \cong X_A \times X_B$—and hence making the very notion of two subsystems well-defined without external labels—is *rectangular completeness*: every pairing of left- and right-profile classes is realized by exactly one state. Section 2 proves this and establishes its minimality (Theorem 2.10). It also shows that closure forces asymmetry (Corollary 2.12): a structurally closed two-subsystem world with more than one state cannot have $c(u,w) = c(w,u)$ for all predicates and states. The same section also proves the corresponding maximality theorem: among locally distinguishable comparison worlds, closure is equivalent to profile-maximality, that is, to the impossibility of any proper profile-determined one-point extension (Theorem 2.19).

Closure as an admissibility boundary. The closure premise sets an admissibility boundary rather than adding dynamics. If the modeled system is closed, a report whose value depends on an external frame, observer label, gauge section, representative choice, or route convention is not a report of that system alone; it is a report of the system together with extra scaffolding. Such structure may be introduced in an open description, but under closed-world admissibility it is allowed only if it is reconstructed from the comparison data themselves. Quotient semantics therefore enters as a consequence rather than as an interpretive convention. The comparison-world form of the admissibility boundary is this: intrinsicity rules out external scaffolding, rectangular completeness is the minimal condition making the two-subsystem product internal, and reports invariant under the resulting redundancy then factor through the orbit quotient.

Thus the diagonal G-action of (1.1) is the canonical output of structural closure in a two-subsystem world, rather than a modeling axiom. The results below derive the diagonal redundancy from the closure criterion itself.

Let G be a group and let X_A, X_B be sets equipped with left G-actions. Set

$$X := X_A \times X_B, \qquad g \cdot (x_A, x_B) := (g \cdot x_A,\ g \cdot x_B). \tag{1.1}$$

The diagonal action encodes *structural redundancy*: points in the same G-orbit represent indistinguishable descriptions of a single physical state.

Motivating example. Take $X_A = X_B = \mathbb{R}^3$ and $G = \mathrm{SO}(3)$ acting by simultaneous rotations. Then (x_A, x_B) and $(g \cdot x_A, g \cdot x_B)$ represent the same state for every $g \in G$. Any SO(3)-invariant function of (x_A, x_B) — such as the relative separation $\|x_A - x_B\|$ — descends to the quotient X/G, whereas the individual directions of x_A and x_B do not. The formalism below abstracts exactly this distinction between quotient-invariant and frame-dependent descriptions.

Nontriviality of the extracted symmetry. Rectangular completeness alone does not force G to have nontrivial orbits. If the comparison predicates rigidly name every state, then $G = \{e_G\}$, $\mathsf{Phys} = X$, and the pure-orbit obstruction is vacuous. The substantive no-attribution results therefore apply to closed comparison worlds whose internally extracted automorphism group has nontrivial orbit motion. This is a condition on the comparison data, not an additional external gauge postulate: when the admitted predicates are invariant under common frame changes, the corresponding common transformations are elements of the extracted group G.

Concrete readings. Relative motion remains observable at quotient level. If the separation, relative orientation, or comparison outcome changes, that change is quotient-level content and is detected by coherent (orbit-invariant) reports. The obstruction is sharper: once two descriptions determine the same quotient history in Phys, no closed-system report can decide which subsystem should be said to have carried the change.

In ordinary relational two-body kinematics, take joint descriptions $(q_A, q_B) \in \mathbb{R}^3 \times \mathbb{R}^3$ and let the Euclidean group act diagonally by changing the common frame. Reports such as $\|q_A - q_B\|$ descend to the quotient; reports such as "the absolute position of A" do not. If two histories have the same pointwise orbit projection, with one keeping A fixed while B varies and the other keeping B fixed while A varies, the theorem implies that quotient data alone do not choose between the two support attributions. Making such a choice is exactly representative selection: one has chosen a frame, origin, gauge, or other section of π.

For an electron–detector pair, the comparison data may record which joint states agree on an outcome, whether one state has the larger value of a comparison quantity, or which states are related by an allowed preparation/readout relation. Outcome correlations are quotient-level reports when they are invariant under the intrinsic relabelings of the comparison world. By contrast, assigning the same invariant change uniquely to "the electron" rather than to "the detector description" requires a representative of the joint orbit: a calibration convention, apparatus frame, or other selector.

For a finite network of comparisons, morphism-level transport is the second enrichment locus. If comparisons transported along two co-terminal routes disagree, the disagreement is measured by a loop defect $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$ for a loop based at p. The two-locus theorem proves that, in the finitary setting of this paper, every non-endpoint-determined refinement has route-dependent transport with nontrivial loop defect, possibly parameterized by representative selection.

Relation to existing work. The question sits in the same foundations territory as relational observables, gauge redundancy, and constrained dynamics in closed systems. Relational quantum mechanics and partial-observable approaches emphasize ways of formulating observable content in generally covariant or closed settings without appeal to an external frame [1, 2, 4]. Work on spacetime diffeomorphisms, Machian relational dynamics, and gauge freedom likewise isolates the distinction between redundant description and physical content [3, 5, 6]. The contribution here is more austere: starting only from binary comparison data and an internal closure criterion, the paper derives the diagonal redundancy, the forced quotient semantics, the obstruction to subsystem attribution, and the two-locus exhaustion theorem for finitary extensions.

Origin of the diagonal action. The setup (1.1) is not an axiom but a theorem. Section 2 derives it canonically from a *comparison world*—a set U of states equipped only with binary comparison predicates, with no topology or geometry assumed—once that world satisfies rectangular completeness. No symmetry, product structure, or background geometry enters the derivation; all structure is extracted from the comparison predicates themselves (Remark 2.3). The same section shows that rectangular completeness is the minimal internal condition producing the product decomposition. Readers who accept (1.1) as given may proceed directly to Section 3.

Closed-world admissibility hypothesis. For the closed-world results, the modeled system is treated under three clauses.

(i) (*Intrinsicity.*) No structure is admissible unless it is internally recoverable from comparison data alone. No background manifold, external observer frame, or auxiliary state-space primitive may be imported by convention.

(ii) (*Comparison completeness.*) The system is closed in the sense of rectangular completeness: every pairing of left- and right-profile classes is realized by exactly one state. This clause is formalized in Section 2; it is the minimal internal condition that forces the canonical product decomposition $U \cong X_A \times X_B$ and the diagonal G-action of (1.1).

(iii) (*Quotient descent.*) Admissible reports are exactly the *coherent* (G-invariant) maps $R : X \to S$, that is, those satisfying

$$R(g \cdot x) = R(x) \qquad \forall g \in G,\ \forall x \in X.$$

Equivalently (Theorem 3.2), admissible reports are precisely the maps descending along the orbit projection

$$\pi : X \longrightarrow \mathsf{Phys} := X/G. \tag{1.2}$$

The three clauses are logically connected. Clause *(i)* rules out external scaffolding. Clause *(ii)* is the minimal internal condition that, given *(i)*, forces the product structure and symmetry group to be canonically determined by the comparison data. Clause *(iii)* gives the semantic form of closed-system admissibility: once admissible content is internal and comparison-based, it is represented by maps that factor through π. Section 4 gives the corresponding operational formulation: for internally generated reports, operational closure is equivalent to quotient factorization.

Main results. First, among locally distinguishable comparison worlds, rectangular completeness is equivalent to profile-maximality against proper profile-determined one-point extensions (Theorem 2.19). The remaining main results are three structural theorems.

(A) Forced quotient semantics (Theorem 3.2, Theorem 4.6). Every admissible closed-system report factors uniquely through $\pi : X \to \mathsf{Phys}$. No internally generated protocol-local report distinguishes orbit-equivalent states; under operational closure this applies to all internally generated reports. Any additional distinguishability requires supplementary structure beyond quotient semantics.

(B) Obstruction to subsystem attribution (Theorem 7.1, Theorem 7.3). For pure orbit motion, every coherent report is constant. When both genuinely A- and B-supported representatives exist, no orbit-projection-invariant rule correctly identifies which subsystem moved.

(C) Two-locus exhaustion (Theorem 9.10). Within the finitary formalism, every functorial extension of quotient semantics on finite comparison networks draws its supplementary structure from one or both of two loci:

1. representative selection via a selector $s_V : V \to X$, and, when chosen uniformly, via a section $s : \mathsf{Phys} \to X$;
2. morphism-level transport data.

Non-endpoint-determined behavior occurs exactly when the transport locus has nontrivial holonomy/loop defect. Moreover, these two loci are categorically complementary and exhaustive: (1) is object-level enrichment, choosing a preferred representative in each orbit, while (2) is morphism-level enrichment, assigning distinct transport to co-terminal paths.

Why the exhaustion matters. The two loci in result *(C)* have familiar analogues in physics. Locus (1), representative selection via a section $s : \mathsf{Phys} \to X$, is analogous to gauge fixing: one chooses a preferred representative in each orbit. Locus (2), morphism-level transport with possible nontrivial holonomy, is analogous to connection-like transport: holonomy measures the failure of path-independence. These analogies motivate the terminology, but they play no role in the proofs. The theorem itself is internal to the present framework: within the finitary setting, no third independent mechanism exists. This is a formal exhaustion claim relative to Definition 9.2, not a claim that all possible physical theories admit only two kinds of supplementary structure. The substantive content is the holonomy characterization: within this formalism, non-endpoint-determined behavior occurs exactly when the transport locus carries nontrivial loop defect.

Theorem 1.1 (Elimination of external labels)**.** *Let $L \neq \varnothing$ be a set of putative external-observer labels, and let*

$$F : L \times X \to S$$

be a map. Assume external-label symmetry*: for every bijection $\sigma : L \to L$,*

$$F(\sigma(\lambda), x) = F(\lambda, x) \qquad \forall \lambda \in L,\ \forall x \in X. \tag{1.3}$$

Then there exists a unique map $R : X \to S$ such that

$$F = R \circ \mathrm{pr}_X, \qquad \mathrm{pr}_X : L \times X \to X. \tag{1.4}$$

If, in addition, each slice $F_\lambda : X \to S$, $F_\lambda(x) := F(\lambda, x)$, is coherent (that is, G-invariant), then R is coherent and there exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ with

$$F = \widetilde{R} \circ \pi \circ \mathrm{pr}_X. \tag{1.5}$$

Proof. Choose $\lambda_0 \in L$ and define $R : X \to S$ by $R(x) := F(\lambda_0, x)$. Let $\lambda \in L$. If $\lambda = \lambda_0$, then $F(\lambda, x) = R(x)$. If $\lambda \neq \lambda_0$, let $\sigma : L \to L$ be the transposition of λ_0 and λ. Then $\sigma(\lambda_0) = \lambda$, so by (1.3),

$$F(\lambda, x) = F(\sigma(\lambda_0), x) = F(\lambda_0, x) = R(x)$$

for all x. Thus $F = R \circ \mathrm{pr}_X$. If $F = R' \circ \mathrm{pr}_X$, then $R'(x) = F(\lambda_0, x) = R(x)$ for all x, so $R' = R$. If every F_λ is coherent, then $R = F_{\lambda_0}$ is coherent. Theorem 3.2 therefore yields a unique $\widetilde{R}$ with $R = \widetilde{R} \circ \pi$, giving (1.5). □

Remark 1.2 (Interpretation of external-label elimination)**.** *Under closure-compatible symmetry, an external label $\lambda \in L$ adds no invariant physical content. Such labels may be introduced formally, but they remain semantically inert unless one abandons closure or the symmetry condition* (1.3).

At the state level, coherent reports factor as $R = \widetilde{R} \circ \pi$ (Theorem 3.2), pure orbit loops project to constants in Phys (Lemma 5.2), and subsystem attribution fails at the quotient level (Theorem 7.1, Corollary 7.2, and Theorem 7.3). At the comparison level, all supplementary data lie in the selector and transport loci, and comparison becomes non-endpoint-determined exactly when the morphism-level transport datum has nontrivial loop defect (Theorem 9.10 and Corollary 9.11).

The analysis is structural throughout. Its primitives are the relational structure $(U, \mathscr{C})$ and its automorphism group (Section 2); its admissibility criterion is internal operational closure; and its main result is a finitary categorical exhaustion theorem about functorial extensions of quotient semantics. No gauge structure, continuous geometry, Hamiltonian, or spacetime background is invoked at any stage.

The analysis treats the two-subsystem case under the stated closure hypotheses. For three or more subsystems, pairwise overlap constraints impose additional coherence conditions whose failures can be expressed as loop or compatibility defects (Section 9, Subsection 9.4). Appendix A gives a separate illustration of the same descent mechanism for reparametrization redundancy.

Notation and conventions

Only notation used across several sections is fixed here; more specialized notation is introduced at first use.

All group actions are left actions, composition is written right-to-left, and the identity element of G is denoted by e_G. Write $X := X_A \times X_B$ for the diagonal G-space, $\pi : X \to \mathsf{Phys} := X/G$ for the orbit projection, and $[x] := \pi(x)$ for the orbit of $x \in X$. For a fixed $T > 0$, a history is a map $\Gamma : [0,T] \to X$, with $\mathscr{H}_T := X^{[0,T]}$. The appendix uses Y for a generic target set and $\mathscr{H}$ for a set of histories, reserving X for the subsystem product in the main text.

The symbol E is reserved for edge sets of finite networks $\mathscr{N} = (V,E)$. Putative external-observer labels are denoted by L, with elements $\lambda \in L$, to avoid conflating external labels with network edges.

Notation for sections and orbit-pinning maps is introduced in Section 8; notation for finite protocols, selectors, transport data, loop defects, and quotient-semantic extensions is introduced in Section 9.

2 Canonical Extraction of the Basic Data

The diagonal G-action introduced in Section 1 is derived from primitive relational data alone. This section gives the derivation; the proofs are elementary consequences of the definitions. Readers who accept the setup of Section 1 as given may proceed directly to Section 3.

2.1 Comparison worlds and intrinsic congruences

Definition 2.1 (Comparison world)**.** A *comparison world* is a relational structure [7] $(U,\mathscr{C})$ where U is a set of states and $\mathscr{C}$ is a set of binary predicates $c : U \times U \to \{0,1\}$. No topology, metric, manifold structure, background geometry, or external reference frame is assumed.

Physical interpretation

A pairwise physical description is represented as a comparison world once its joint state set and relevant binary comparisons are specified. In such a representation, U is the set of joint states of the pair, and the predicates $\mathscr{C}$ are the binary comparisons admitted by the description: whether one joint state has greater total energy than another, whether two states agree on a given observable, or whether one is accessible from another by a given process.

Three representative instances:

- *Two gravitating bodies.* U is the set of joint configurations $(q_A, q_B) \in \mathbb{R}^3 \times \mathbb{R}^3$; a predicate $c(u,v)$ records, for instance, whether state u has greater separation $\|q_A - q_B\|$ than state v.
- *Electron and detector.* U is the set of joint states of the electron–apparatus pair; a predicate records whether one joint state registers a given outcome comparison.
- *Two charged particles.* U is the set of joint field configurations; predicates record energy, momentum, or charge comparisons between states.

In the intended two-subsystem reading, the left profile $\mathsf{L}(u)$ records how the joint state u appears in the first comparison slot, and $\mathsf{R}(u)$ records how it appears in the second. The intrinsic congruence α partitions states by their first-slot profile: $u\,\alpha\,v$ means that u and v have identical outgoing comparison profiles. Similarly, β partitions states by incoming comparison profiles. The symmetry group G is the relabeling symmetry of the comparison data themselves—extracted from the predicates, not postulated.

The comparison world is the relational skeleton of a pairwise physical description prior to the addition of any specific dynamical law, Hamiltonian, metric, or background geometry. The structural theorems proved below hold for descriptions of the above kind that satisfy the stated closure hypotheses; no further physical structure is used in the proofs.

Remark 2.2 (Comparison predicates are not measurements)**.** *The predicates $c \in \mathscr{C}$ are binary relations on the state set U in the sense of relational model theory [7]. They are not measurement acts, observer operations, or information-transfer events. Thus $c(u,v) = 1$ says only that the states u and v stand in the specified relation; it does not say that a physical process establishes, transmits, or detects that relation. No observer, apparatus, dynamics, or scale is built into the formalism. Applicability is controlled by the structural hypotheses, not by physical scale: whenever a comparison world satisfies rectangular completeness, the structural theorems of the paper, including the later two-locus theorem, apply to that pairwise relational setting.*

Remark 2.3 (No additional structure assumed)**.** *No symmetry, topology, product structure, or background geometry is assumed. Every structure appearing in the derivation below is extracted from the comparison predicates themselves. The predicates* $c : U \times U \to \{0,1\}$ *are not assumed to be symmetric* ($c(u,v) = c(v,u)$) *or reflexive* ($c(u,u) = 1$)*. This is essential: in the symmetric case* ($c(u,v) = c(v,u)$ *for all* c,u,v)*, one has* $\mathsf{L}(u) = \mathsf{R}(u)$ *for all* u*, whence* $\alpha = \beta$ *and* $X_A = X_B$*, so the two congruences collapse to one and the canonical product decomposition degenerates. As Corollary 2.12 shows, a nontrivial symmetric comparison world cannot be rectangularly complete: the canonical factor map lands on the diagonal of* $X_A \times X_A$ *rather than on the full product. Thus a genuine two-subsystem decomposition in this framework requires non-symmetric comparison data.*

Definition 2.4 (Intrinsic symmetry group)**.** The *intrinsic symmetry group* of $(U,\mathscr{C})$ is its automorphism group in the sense of [7]:

$$G \;:=\; \mathrm{Aut}(U,\mathscr{C}) \;=\; \big\{\phi \in \mathrm{Bij}(U) : c(\phi(u),\phi(v)) = c(u,v) \text{ for all } c \in \mathscr{C},\ u,v \in U\big\}.$$

For each $u \in U$ define the *left profile* $\mathsf{L}(u)(c,w) := c(u,w)$ and the *right profile* $\mathsf{R}(u)(c,w) := c(w,u)$, both as functions $\mathscr{C} \times U \to \{0,1\}$. Define equivalence relations α and β on U by

$$u\,\alpha\,v \iff \mathsf{L}(u) = \mathsf{L}(v), \qquad u\,\beta\,v \iff \mathsf{R}(u) = \mathsf{R}(v). \tag{2.1}$$

These are the *intrinsic congruences* of $(U,\mathscr{C})$. Set $X_A := U/\alpha$, $X_B := U/\beta$, and define the *canonical factor map*

$$\Theta : U \to X_A \times X_B, \qquad \Theta(u) := \big([u]_\alpha,\,[u]_\beta\big). \tag{2.2}$$

Lemma 2.5 (The intrinsic congruences are G-invariant)**.** *Both* α *and* β *are preserved by every* $g \in G$*. Consequently* G *acts on* X_A *by* $g \cdot [u]_\alpha := [g(u)]_\alpha$ *and on* X_B *by* $g \cdot [u]_\beta := [g(u)]_\beta$*.*

Proof. Suppose $u\,\alpha\,v$, that is, $c(u,w) = c(v,w)$ for all $c \in \mathscr{C}$ and $w \in U$. Fix $g \in G$ and $c \in \mathscr{C}$, $w' \in U$. Write $w = g^{-1}(w')$; since g preserves c,

$$c(g(u),w') = c(u,w) = c(v,w) = c(g(v),w').$$

Hence $g(u)\,\alpha\,g(v)$, so $g \cdot [u]_\alpha = [g(u)]_\alpha$ is well-defined. The argument for β is identical. □

2.2 Rectangular completeness and the diagonal action theorem

Definition 2.6 (Rectangular completeness)**.** The comparison world $(U,\mathscr{C})$ is *rectangularly complete* if for every $A \in X_A$ and every $B \in X_B$ there exists a *unique* $u \in U$ with $[u]_\alpha = A$ and $[u]_\beta = B$.

Remark 2.7 (Internal character of rectangular completeness)**.** *Rectangular completeness is a condition on the internal relational structure of* $(U,\mathscr{C})$ *alone. It asserts that every pairing of a left-profile class and a right-profile class is realized by exactly one state. No external observer, boundary, or ambient space is referenced. In the present framework it is the internal closure condition: there are no missing profile pairs in the canonical product extracted from the comparison data.*

Theorem 2.8 (Diagonal action theorem)**.** *Suppose* $(U,\mathscr{C})$ *is rectangularly complete. Then:*

(i) The canonical factor map $\Theta : U \to X_A \times X_B$ *is bijective; hence* $U \cong X_A \times X_B$ *canonically.*

(ii) $G = \mathrm{Aut}(U,\mathscr{C})$ *acts* diagonally *on* $X := X_A \times X_B$*:*

$$g \cdot (x_A, x_B) \;:=\; (g \cdot x_A,\; g \cdot x_B).$$

(iii) The product decomposition, the diagonal G-action, and the orbit quotient $\pi : X \to \mathsf{Phys} := X/G$ *are all canonically determined by* $(U,\mathscr{C})$*.*

Proof. (i) Bijectivity of Θ*. Surjectivity:* Given $(A,B) \in X_A \times X_B$, the existence clause of rectangular completeness provides $u \in U$ with $[u]_\alpha = A$ and $[u]_\beta = B$, so $\Theta(u) = (A,B)$. *Injectivity:* If $\Theta(u) = \Theta(v)$, then $[u]_\alpha = [v]_\alpha$ and $[u]_\beta = [v]_\beta$. The uniqueness clause of rectangular completeness applied to the pair $(A,B) = ([u]_\alpha,\,[u]_\beta)$ gives $u = v$.

(ii) Diagonality. By Lemma 2.5, G acts on both X_A and X_B. For $u \in U$ and $g \in G$,

$$\Theta(g \cdot u) = \big([g(u)]_\alpha,\,[g(u)]_\beta\big) = \big(g \cdot [u]_\alpha,\; g \cdot [u]_\beta\big),$$

so Θ intertwines the G-action on U with the coordinatewise action on $X_A \times X_B$. The coordinatewise action is diagonal by definition.

(iii) Once Θ is a G-equivariant bijection, the product structure $X = X_A \times X_B$, the diagonal action, and the projection $\pi : X \to \mathsf{Phys} := X/G$ are canonically induced. No auxiliary choice is made. $\square$

Remark 2.9 (Nontriviality of the diagonal action)**.** *Theorem 2.8 derives the diagonal action from the comparison data; it does not assert that the action is nontrivial for every rectangularly complete comparison world. If the predicates rigidly distinguish all states, then $G = \{e_G\}$ and $\mathsf{Phys} = X$. The obstruction results become substantive when the comparison predicates leave a nontrivial common relabeling symmetry. For example, if a concrete two-body comparison world carries a simultaneous* $\mathrm{SO}(3)$*-action on its states and every admitted predicate is invariant under that action, then the image of this action lies in G. Thus separation-type or rotation-invariant predicates can witness a nontrivial diagonal symmetry. The full extracted group, however, is always the automorphism group of the given predicate structure. When an application is meant to extract a particular rotation group exactly, that identification is a property of the chosen predicate structure rather than of rotation invariance alone.*

Theorem 2.10 (Rectangular completeness and bijectivity)**.** *For a comparison world $(U, \mathscr{C})$, rectangular completeness is equivalent to bijectivity of the canonical factor map $\Theta : U \to X_A \times X_B$. Consequently, rectangular completeness is the minimal internal condition that forces this canonical product decomposition: any internal property P that ensures Θ is bijective for every comparison world satisfying P implies rectangular completeness.*

Proof. If $(U, \mathscr{C})$ is rectangularly complete, then the proof of Theorem 2.8*(i)* shows that Θ is bijective. Conversely, suppose Θ is bijective. *Existence clause:* surjectivity of Θ gives, for every $(A,B) \in X_A \times X_B$, some u with $[u]_\alpha = A$ and $[u]_\beta = B$. *Uniqueness clause:* injectivity of Θ gives: if $[u]_\alpha = [v]_\alpha$ and $[u]_\beta = [v]_\beta$ then $u = v$. Together these are exactly the statement of rectangular completeness. For the final claim, if P forces Θ to be bijective, then every comparison world satisfying P is rectangularly complete by the equivalence just proved. $\square$

Remark 2.11 (Minimality is a characterization, not a bound)**.** *Theorem 2.10 identifies rectangular completeness as exactly the condition that makes the canonical factor map bijective, and hence as exactly the condition for the canonical product decomposition. This is the precise sense in which closure, as defined here, is the threshold condition for a two-subsystem universe to admit the canonical decomposition extracted from its own comparison data.*

Corollary 2.12 (Closure forces asymmetry)**.** *If $(U, \mathscr{C})$ is rectangularly complete and $|U| > 1$, then the comparison world is not symmetric: there exist $c \in \mathscr{C}$ and $u, w \in U$ such that $c(u,w) \neq c(w,u)$.*

Proof. Suppose for contradiction that $c(u,w) = c(w,u)$ for all $c \in \mathscr{C}$ and all $u, w \in U$. Then for each $u \in U$, all $c \in \mathscr{C}$, and all $w \in U$,

$$\mathsf{L}(u)(c,w) = c(u,w) = c(w,u) = \mathsf{R}(u)(c,w),$$

so $\mathsf{L}(u) = \mathsf{R}(u)$. Therefore $u\,\alpha\,v \iff u\,\beta\,v$, that is, $\alpha = \beta$ and $X_A = X_B = U/\alpha$. Since rectangular completeness makes Θ bijective, the map $u \mapsto [u]_\alpha$ is injective: if $[u]_\alpha = [v]_\alpha$, then also $[u]_\beta = [v]_\beta$, hence $\Theta(u) = \Theta(v)$ and therefore $u = v$. Thus X_A has at least two elements. The canonical factor map becomes $\Theta(u) = ([u]_\alpha, [u]_\alpha)$, so the image of Θ is the diagonal $\Delta \subseteq X_A \times X_A$. Choose distinct $A, B \in X_A$. Then $(A,B) \in X_A \times X_A$ but $(A,B) \notin \Delta$, so Θ is not surjective, contradicting rectangular completeness. $\square$

Remark 2.13 (Structural content of closure-forced asymmetry)**.** *Corollary 2.12 makes asymmetry a structural consequence rather than a convention about which comparison predicates to include. A nontrivial comparison world closed in the sense of rectangular completeness cannot have fully symmetric comparison data. If $c(u,w) = c(w,u)$ for every predicate and every pair of states, the two intrinsic congruences collapse and the canonical factor map cannot cover the full product. Thus, in this framework, a genuine two-subsystem decomposition requires directional comparison data.*

2.3 Closure as profile-maximality

This subsection proves the complementary maximality statement to Theorem 2.10. Once one restricts to locally distinguishable comparison worlds and allows only intrinsic, profile-determined one-point extensions, rectangular completeness is both minimal and maximal.

Definition 2.14 (Local distinguishability and profile-pair defects)**.** A comparison world $(U,\mathscr{C})$ is *locally distinguishable* if the canonical factor map $\Theta : U \to X_A \times X_B$ is injective. Equivalently, whenever two states have the same left-profile class and the same right-profile class, they are equal.

A pair $(A,B) \in X_A \times X_B$ not realized by any state $u \in U$ is called a *profile-pair defect*.

Definition 2.15 (Profile-determined one-point extension)**.** Let $(U,\mathscr{C})$ be a comparison world. A *profile-determined one-point extension* of $(U,\mathscr{C})$ is a comparison world $(U^*,\mathscr{C}^*)$ with

$$U^* = U \cup \{u^*\}, \qquad u^* \notin U,$$

and $\mathscr{C}^* = \{c^* : c \in \mathscr{C}\}$, where each c^* extends c on $U \times U$ and there exist classes $A \in X_A$ and $B \in X_B$ such that, for every $v \in U$,

$$c^*(u^*,v) := c(a,v), \tag{2.3}$$

$$c^*(v,u^*) := c(v,b), \tag{2.4}$$

$$c^*(u^*,u^*) := c(a,b), \tag{2.5}$$

for any representatives $a \in A$ and $b \in B$. The first right-hand side is independent of a because A is an α-class; the second is independent of b because B is a β-class. The diagonal value is independent of both representatives: if $a,a' \in A$ and $b,b' \in B$, then $c(a,b) = c(a',b)$ by $a\,\alpha\,a'$ and $c(a',b) = c(a',b')$ by $b\,\beta\,b'$. The pair (A,B) is called the *assigned profile pair* of u^*.

Definition 2.16 (Profile-maximality)**.** A locally distinguishable comparison world is *profile-maximal* if it admits no proper locally distinguishable profile-determined one-point extension.

Lemma 2.17 (A defect can be adjoined without losing local distinguishability)**.** *Let $(U,\mathscr{C})$ be locally distinguishable, and let $(A,B) \in X_A \times X_B$ be a profile-pair defect. Then there exists a locally distinguishable profile-determined one-point extension of $(U,\mathscr{C})$ whose assigned profile pair is (A,B).*

Proof. Choose representatives $a \in A$ and $b \in B$, and define $(U^*,\mathscr{C}^*)$ by equations (2.3)–(2.5). Let α^* and β^* be the intrinsic congruences of $(U^*,\mathscr{C}^*)$.

First, for $v \in U$ one has

$$u^*\,\alpha^*\,v \quad\Longleftrightarrow\quad v \in A.$$

Indeed, if $u^*\,\alpha^*\,v$, then for every $c \in \mathscr{C}$ and every $w \in U$,

$$c(a,w) = c^*(u^*,w) = c^*(v,w) = c(v,w),$$

so $v \in A$. Conversely, if $v \in A$, then for $w \in U$ one has $c^*(u^*,w) = c(a,w) = c(v,w) = c^*(v,w)$, while for $w = u^*$,

$$c^*(u^*,u^*) = c(a,b) = c(v,b) = c^*(v,u^*),$$

because $v\,\alpha\,a$. Thus $u^*\,\alpha^*\,v$. The same argument, with left and right interchanged, shows that for $v \in U$,

$$u^*\,\beta^*\,v \quad\Longleftrightarrow\quad v \in B.$$

Now let $v,w \in U$ be distinct. Since $(U,\mathscr{C})$ is locally distinguishable, v and w differ in at least one old left or right profile entry, witnessed by some predicate in $\mathscr{C}$ and some comparison state in U. Because each c^* restricts to c on $U \times U$, the same witness remains valid in $(U^*,\mathscr{C}^*)$. Hence distinct states already in U remain locally distinguishable.

Finally, let $v \in U$. If u^* had the same left and right profile classes as v in $(U^*,\mathscr{C}^*)$, then $u^*\,\alpha^*\,v$ and $u^*\,\beta^*\,v$, hence $v \in A$ and $v \in B$. That would make v a realization of the pair (A,B), contradicting that (A,B) is a profile-pair defect. Therefore u^* is distinguishable from every state of U.

So $(U^*,\mathscr{C}^*)$ is locally distinguishable. □

Lemma 2.18 (Closure blocks profile-determined adjunctions)**.** *If $(U,\mathscr{C})$ is rectangularly complete, then every profile-determined one-point extension of $(U,\mathscr{C})$ fails to be locally distinguishable.*

Proof. Let $(U^*,\mathscr{C}^*)$ be a profile-determined one-point extension with assigned profile pair (A,B). Choose representatives $a \in A$ and $b \in B$ as in Definition 2.15. By rectangular completeness there exists a unique $w \in U$ with $[w]_\alpha = A$ and $[w]_\beta = B$. It remains to show that u^* and w have the same left and right profile classes in $(U^*,\mathscr{C}^*)$.

For every $v \in U$ and every $c \in \mathscr{C}$,

$$c^*(u^*, v) = c(a, v) = c(w, v)$$

because $w \in A$, and

$$c^*(v, u^*) = c(v, b) = c(v, w)$$

because $w \in B$. For the comparison with the new point itself,

$$c^*(u^*, u^*) = c(a, b) = c(w, b) = c^*(w, u^*)$$

because $w \in A$, and likewise

$$c^*(u^*, u^*) = c(a, b) = c(a, w) = c^*(u^*, w)$$

because $w \in B$. Thus u^* and w agree against every comparison state in U^* on both the left and right. Hence $u^* \, \alpha^* \, w$ and $u^* \, \beta^* \, w$. Since $u^* \neq w$, local distinguishability fails. □

Theorem 2.19 (Closure $\Longleftrightarrow$ profile-maximality)**.** *Within the class of locally distinguishable comparison worlds, rectangular completeness is equivalent to profile-maximality.*

Proof. Suppose first that $(U, \mathscr{C})$ is rectangularly complete. Then Lemma 2.18 shows that no proper profile-determined one-point extension can remain locally distinguishable. Hence $(U, \mathscr{C})$ is profile-maximal.

Conversely, suppose $(U, \mathscr{C})$ is locally distinguishable and profile-maximal. If rectangular completeness failed, there would exist a profile-pair defect $(A, B) \in X_A \times X_B$. Lemma 2.17 would then produce a proper locally distinguishable profile-determined one-point extension, contradicting profile-maximality. Therefore $(U, \mathscr{C})$ is rectangularly complete. □

Remark 2.20 (Scope of the closure–maximality equivalence)**.** *Theorem 2.19 identifies the intrinsic maximality notion of the present framework. The extension is required to be* profile-determined*: the only datum used in adjoining the new state is a left-profile class together with a right-profile class already available in $(U, \mathscr{C})$. Allowing unrestricted new comparison values among newly adjoined states would import extra structure not fixed by the original comparison world. Within the intrinsic comparison formalism, this theorem gives the exact sense in which a closed world is already maximal.*

2.4 Semantic extraction from comparison closure

Theorems 2.8, 2.10, and 2.19 show that the data introduced in Section 1—the product $X = X_A \times X_B$, the group $G = \mathrm{Aut}(U, \mathscr{C})$, and the projection $\pi : X \to \mathsf{Phys} := X/G$—are the unique canonical output of any comparison world satisfying rectangular completeness. They also show that rectangular completeness is both the minimal condition yielding these data and, among locally distinguishable worlds, the intrinsic maximality condition forbidding further profile-determined one-point extensions.

The subsequent sections take this output as fixed and determine: (a) which state reports survive as admissible closed-system content, namely the coherent maps that factor through π; (b) what no coherent report can distinguish, namely subsystem attribution along pure orbit loops; and (c) where any supplementary structure must enter, namely the two exhaustive loci isolated below.

3 Diagonal Redundancy and Descent

This section isolates the descent mechanism: once admissible reports respect the redundancy, all admissible state-level content factors uniquely through the orbit quotient.

Let G be a group acting on the set X, and let

$$\pi : X \longrightarrow \mathsf{Phys} := X/G \tag{3.1}$$

denote the orbit projection.

3.1 Coherent reports

Definition 3.1 (Report and coherence)**.** A *report* is a function $R : X \to S$. It is *coherent* (equivalently, G-invariant) if

$$R(g \cdot x) = R(x) \qquad \forall g \in G, \, \forall x \in X. \tag{3.2}$$

3.2 Universal property of the orbit quotient

Theorem 3.2 (Forced descent and the universal property of Phys)**.** *Let $R : X \to S$ be coherent. Then there exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ with*

$$R = \widetilde{R} \circ \pi. \tag{3.3}$$

Equivalently, (Phys, π) *is the universal object through which every G-invariant map factors uniquely [8].*

Proof. Define $\widetilde{R}([x]) := R(x)$. If $[x] = [y]$ then $y = g \cdot x$, so $R(y) = R(x)$ by coherence; $\widetilde{R}$ is well-defined. The identity $(\widetilde{R} \circ \pi)(x) = \widetilde{R}([x]) = R(x)$ proves factorization (3.3). Uniqueness follows because if $f \circ \pi = R$, then $f([x]) = R(x) = \widetilde{R}([x])$. □

3.3 Equivalent formulations

Corollary 3.3 (Equivalent formulations of coherence)**.** *For $R : X \to S$, the following are equivalent:*

(i) *R is coherent.*

(ii) *R is constant on G-orbits.*

(iii) *There exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ with $R = \widetilde{R} \circ \pi$.*

Proof. (i)⇒(ii): If $y = g \cdot x$ then $R(y) = R(x)$. (ii)⇒(i): x and $g \cdot x$ lie in the same orbit. (i)⇒(iii): Theorem 3.2. (iii)⇒(ii): $\pi(x) = \pi(y)$ gives $R(x) = R(y)$. □

3.4 Categorical formulation

Remark 3.4 (Terminality of the orbit quotient)**.** *Let $\mathsf{Inv}(X)$ have objects (S, R) (R coherent) and morphisms $(S, R) \to (S', R')$ given by $f : S' \to S$ with $R = f \circ R'$. Then (Phys, π) is a terminal object of $\mathsf{Inv}(X)$ [8].*

Therefore orbit-level data are complete for admissible state reports. Any further distinguishability requires additional non-quotient structure.

4 Internal Closure and Indistinguishability

This section formulates closure as an internal operational criterion and proves that, for internally generated reports, it is equivalent to quotient-semantic factorization through $\pi : X \to \mathsf{Phys}$. Thus the closure premise used elsewhere in the paper can be read either as a structural admissibility premise (Section 1) or operationally.

4.1 Internal reports and protocol locality

Definition 4.1 (Internally generated report)**.** A report $R : X \to S$ is *internally generated* if there exist:

(i) a finite protocol $(\mathscr{N}, \gamma)$ in the sense of Definition 9.1, with vertex set V;

(ii) a distinguished readout vertex $v_* \in V$ and a realization map $\Lambda : X \to X^V$, written $x \mapsto \lambda_x$, such that $\lambda_x(v_*) = x$ for all $x \in X$;

(iii) a set Q of elementary comparison values and a kernel $\kappa : X \times X \to Q$;

(iv) a relabeling-invariant evaluator $\mathsf{Ev}_{\mathscr{N},\gamma} : \mathscr{D}_{\mathscr{N},\gamma}(Q) \to S$, where $\mathscr{D}_{\mathscr{N},\gamma}(Q)$ denotes finite Q-labeled protocol realizations;

such that, for every $x \in X$,

$$R(x) = \mathsf{Ev}_{\mathscr{N},\gamma}\big(\mathscr{D}_{\mathscr{N},\gamma}(\lambda_x; \kappa)\big).$$

Here relabeling-invariance means invariance under isomorphisms of the finite protocol data that preserve the distinguished path and readout vertex. Thus the evaluator may depend on the internal pattern of comparison values, but not on arbitrary names assigned to auxiliary protocol vertices. Let $\mathscr{O}_{\text{int}}$ denote the class of all internally generated reports.

Definition 4.2 (Protocol-local report)**.** An internally generated report is *protocol-local* if, for every witness $(\mathscr{N}, \gamma, V, v_*, \Lambda, \kappa, \mathsf{Ev}_{\mathscr{N},\gamma})$ realizing it in the sense of Definition 4.1, whenever $\pi(x) = \pi(y)$ one has

$$\mathsf{Ev}_{\mathscr{N},\gamma}\big(\mathscr{D}_{\mathscr{N},\gamma}(\lambda_x; \kappa)\big) = \mathsf{Ev}_{\mathscr{N},\gamma}\big(\mathscr{D}_{\mathscr{N},\gamma}(\lambda_y; \kappa)\big).$$

For a given witness, a sufficient condition is the existence of a descended kernel $\overline{\kappa} : \mathsf{Phys} \times \mathsf{Phys} \to Q$ with $\kappa(x_i, x_j) = \overline{\kappa}(\pi(x_i), \pi(x_j))$.

Remark 4.3 (Universality with respect to witnesses)**.** *Protocol locality is a property of the report R itself, quantified over its internal-generation witnesses. No additional faithfulness assumption on the comparison data is imposed.*

Definition 4.4 (Operational closure)**.** The system is *operationally closed* if every internally generated report is protocol-local.

4.2 Protocol locality and coherence

Lemma 4.5 (Protocol locality implies coherence)**.** *Let $R : X \to S$ be internally generated and protocol-local. Then $R(g \cdot x) = R(x)$ for all $g \in G$, $x \in X$. Hence R is coherent.*

Proof. Choose a witness $(\mathscr{N}, \gamma, V, v_*, \Lambda, \kappa, \mathsf{Ev}_{\mathscr{N},\gamma})$ for R. Fix $g \in G$ and $x \in X$. Since $\pi(g \cdot x) = \pi(x)$, protocol locality for this witness gives

$$\mathsf{Ev}_{\mathscr{N},\gamma}\big(\mathscr{D}_{\mathscr{N},\gamma}(\lambda_{g \cdot x}; \kappa)\big) = \mathsf{Ev}_{\mathscr{N},\gamma}\big(\mathscr{D}_{\mathscr{N},\gamma}(\lambda_x; \kappa)\big).$$

No equivariance assumption on Λ is used here: protocol locality is applied directly to the pair of states $g \cdot x$ and x, whose orbit projections coincide. By Definition 4.1, the left side is $R(g \cdot x)$ and the right side is $R(x)$. □

4.3 Internal closure and quotient equivalence

Theorem 4.6 (Internal closure implies descent)**.** *Every internally generated protocol-local report $R : X \to S$ is coherent, and there exists a unique $\widetilde{R} : \mathsf{Phys} \to S$ such that $R = \widetilde{R} \circ \pi$.*

Proof. Lemma 4.5 gives coherence. Theorem 3.2 gives unique factorization. □

Corollary 4.7 (Internal indistinguishability)**.** *If R is internally generated and protocol-local, and if $\pi(x) = \pi(y)$, then $R(x) = R(y)$. Under operational closure, no internally generated report distinguishes orbit-equivalent states.*

Proof. By Theorem 4.6, $R = \widetilde{R} \circ \pi$ for a unique $\widetilde{R} : \mathsf{Phys} \to S$. Hence $\pi(x) = \pi(y)$ implies $R(x) = \widetilde{R}(\pi(x)) = \widetilde{R}(\pi(y)) = R(y)$. Under operational closure, every internally generated report is protocol-local, so the same argument applies to all internally generated reports. □

Corollary 4.8 (No internal representative distinction)**.** *Under operational closure, any internally generated admissible report is constant on each orbit. Therefore a preferred representative inside a nontrivial orbit cannot be distinguished by quotient-level content alone; it requires supplementary selector data.*

Proof. The first statement is Corollary 4.7. If an orbit is nontrivial and a rule distinguishes one of its representatives by report values, then that report is not constant on the orbit. Such a rule therefore cannot be supplied by quotient-level content alone. A selector is precisely the additional object-level datum that chooses representatives in the fibers of π. □

Corollary 4.9 (Internal closure implies loop-level indistinguishability)**.** *Under operational closure, every internally generated report is constant on pure diagonal orbit loops and cannot distinguish a loop from its twin (Theorem 6.1).*

Proof. Operational closure gives protocol locality, hence coherence by Theorem 4.6. Apply Theorem 7.1. □

Theorem 4.10 (Operational closure–quotient equivalence)**.** *For internally generated reports, the following are equivalent:*

(i) *the system is operationally closed;*

(ii) *every* $R \in \mathscr{O}_{\text{int}}$ *factors through* π.

Proof. (i)⇒(ii): under operational closure, every internally generated report is protocol-local, so Theorem 4.6 applies. (ii)⇒(i): if $R = \widetilde{R} \circ \pi$ and $\pi(x) = \pi(y)$, then $R(x) = R(y)$. For any witness representation of R in Definition 4.1, writing both sides through that witness yields equal evaluator outputs, so R is protocol-local. □

Corollary 4.11 (Universe-level admissibility criterion)**.** *Assume the modeled system is the universe and adopt operational closure for internally generated reports. Then an internally generated report is admissible if and only if it is of the form* $R = \widetilde{R} \circ \pi$ *for some* $\widetilde{R} : \mathsf{Phys} \to S$*. In particular, if* $\pi(x) = \pi(y)$ *then no internally generated admissible report distinguishes* x *from* y*.*

Proof. By Theorem 4.10, operational closure is equivalent to factorization through π for internally generated reports. The indistinguishability claim is Corollary 4.7. □

Remark 4.12 (Relation to the two-locus theorem)**.** *Theorem 4.10 gives*

$$\textit{operational closure for } \mathscr{O}_{\text{int}} \iff \textit{quotient factorization for } \mathscr{O}_{\text{int}}.$$

Combined with Theorem 9.10, this yields the chain: internal indistinguishability ⇒ quotient semantics ⇒ two-locus obstruction. This chain appears in the closure-rigidity formulation of Section 10 as the internal-admissibility elimination step (Lemma 10.5).

5 Pure Orbit Loops

Fix $T > 0$ and regard $[0,T]$ simply as a set with distinguished points $0, T$; no topology is assumed.

A *history* is $\Gamma : [0,T] \to X$, and $\mathscr{H}_T := X^{[0,T]}$ denotes the set of histories.

5.1 Definitions and support conditions

Definition 5.1 (Pure diagonal orbit loop)**.** A *pure diagonal orbit loop* is a history $\Gamma : [0,T] \to X$ for which there exists $g : [0,T] \to G$ with $g(0) = g(T) = e_G$ and

$$\Gamma(t) = g(t) \cdot \Gamma(0) \qquad \forall t. \tag{5.1}$$

In particular, $\Gamma(T) = \Gamma(0)$.

A history Γ_A is *A-supported* if $\Gamma_A(t) = (x_A(t), x_B^*)$ for some function $x_A : [0,T] \to X_A$ and fixed $x_B^* \in X_B$. Similarly, a history Γ_B is *B-supported* if $\Gamma_B(t) = (x_A^*, x_B(t))$ for fixed $x_A^* \in X_A$ and some $x_B : [0,T] \to X_B$.

A pure diagonal orbit loop *admits an A-supported representative* if there exists an A-supported history Γ_A with

$$\pi(\Gamma_A(t)) = \pi(\Gamma(t)) \qquad \forall t. \tag{5.2}$$

Similarly for *B-supported representatives*.

5.2 Single-orbit property

Lemma 5.2 (Single-orbit property)**.** *If* Γ *is a pure diagonal orbit loop, then* $\pi(\Gamma(t)) = \pi(\Gamma(0))$ *for all* t*.*

Proof. $\Gamma(t) = g(t) \cdot \Gamma(0)$ lies in the G-orbit of $\Gamma(0)$; π is constant on orbits. □

6 Orbit-Equivalent Twin Histories

Theorem 6.1 (Construction of a twin history)**.** *Let* Γ *be a pure diagonal orbit loop with* $\Gamma(t) = g(t) \cdot \Gamma(0)$*, admitting an A-supported representative* $\Gamma_A(t) = (x_A(t), x_B^*)$*. Define*

$$\widetilde{\Gamma}(t) := g(t)^{-1} \cdot \Gamma_A(t). \tag{6.1}$$

Then the following hold:

(i) $\Gamma_A(t) = g(t) \cdot \widetilde{\Gamma}(t)$*;*

(ii) $\pi(\Gamma(t)) = \pi(\widetilde{\Gamma}(t))$*;*

(iii) *the constant history* $\widetilde{\Gamma}_B(t) := \Gamma(0) = (x_A^0, x_B^0)$ *is B-supported in the sense of Definition 5.1 (not necessarily genuinely B-supported) and satisfies* $\pi(\Gamma_A(t)) = \pi(\widetilde{\Gamma}_B(t))$.

Proof. (i) $g(t) \cdot (g(t)^{-1} \cdot \Gamma_A(t)) = \Gamma_A(t)$. (ii) $\widetilde{\Gamma}(t)$ and $\Gamma_A(t)$ lie in the same orbit, while $\pi(\Gamma_A(t)) = \pi(\Gamma(t))$ by definition of representative. (iii) Define $\widetilde{\Gamma}_B(t) := \Gamma(0) = (x_A^0, x_B^0)$; this is B-supported in the sense that its A-component is the constant x_A^0, though it need not be genuinely B-supported. By Lemma 5.2 and (5.2), $\pi(\Gamma_A(t)) = \pi(\Gamma(t)) = \pi(\widetilde{\Gamma}_B(t))$. □

Remark 6.2 (Twin histories and pure orbit loops)**.** *The construction above always produces a history* $\widetilde{\Gamma}$ *with the same orbit projection as* Γ*, but it need not itself be a pure diagonal orbit loop unless additional compatibility conditions are imposed on* Γ_A.

7 Obstruction to Subsystem Attribution

Theorem 7.1 (Failure of invariant subsystem attribution)**.** *Let* $R : X \to S$ *be coherent and let* Γ *be a pure diagonal orbit loop. Then* $R \circ \Gamma$ *is constant. If* Γ *admits an A-supported representative and* $\widetilde{\Gamma}$ *is the associated twin history from Theorem 6.1,*

$$R(\Gamma(t)) = R(\widetilde{\Gamma}(t)) \qquad \forall t. \tag{7.1}$$

Proof. By Theorem 3.2, $R = \widetilde{R} \circ \pi$. Lemma 5.2 gives $\pi(\Gamma(t)) = \pi(\Gamma(0))$, so $R(\Gamma(t)) = \widetilde{R}(\pi(\Gamma(0)))$ is constant. Theorem 6.1(ii) gives $\pi(\widetilde{\Gamma}(t)) = \pi(\Gamma(t))$, so $R(\widetilde{\Gamma}(t)) = R(\Gamma(t))$. □

Corollary 7.2 (No subsystem attribution at quotient level)**.** *Every coherent report is constant along any pure diagonal orbit loop. In particular, when orbit-projection-equivalent A- and B-supported descriptions are both present, orbit-level data alone do not determine which support reading has been chosen. This is illustrated in Figure 1.*

Proof. The constancy claim is Theorem 7.1. If two descriptions have the same orbit projection pointwise, then every coherent report has the same value on both descriptions by Theorem 3.2. Hence quotient-level data cannot distinguish the two support attributions. □

Theorem 7.3 (Rigidity of orbit-projection-invariant support attribution)**.** *Assume there exists a pure diagonal orbit loop* Γ *admitting both a genuinely A-supported representative* $\Gamma_A(t) = (x_A(t), x_B^*)$ *with* x_A *non-constant, and a genuinely B-supported representative* $\Gamma_B(t) = (x_A^*, x_B(t))$ *with* x_B *non-constant. Then there is no* $f : \mathscr{H}_T \to \{A, B\}$ *satisfying both:*

(i) *(*Orbit-Projection Invariance*)* $\pi \circ H_1 = \pi \circ H_2$ *pointwise* $\Rightarrow f(H_1) = f(H_2)$.

(ii) *(*Support Correctness*) every genuinely A-supported history satisfies* $f(H) = A$*, and every genuinely B-supported history satisfies* $f(H) = B$*, where "genuinely" means that the moving component is non-constant.*

Proof. Support correctness gives $f(\Gamma_A) = A$ and $f(\Gamma_B) = B$. By the definition of representative, the two assumed representatives satisfy

$$\pi(\Gamma_A(t)) = \pi(\Gamma(t)) = \pi(\Gamma_B(t)) \qquad \forall t.$$

Thus $\pi \circ \Gamma_A = \pi \circ \Gamma_B$ pointwise, so orbit-projection invariance gives $f(\Gamma_A) = f(\Gamma_B)$. Hence $A = B$, a contradiction. □

8 Orbit-Preserving Maps and Quotient Invisibility

8.1 Orbit-preserving maps

Definition 8.1 (Orbit-preserving map)**.** $\Theta_X : X \to X$ is *orbit-preserving* if $\pi \circ \Theta_X = \pi$.

Lemma 8.2 (Identity on the quotient)**.** *If* Θ_X *is orbit-preserving, then* $\pi(\Theta_X(x)) = \pi(x)$ *for every* $x \in X$. *Hence* Θ_X *induces* $\mathrm{id}_{\mathsf{Phys}}$.

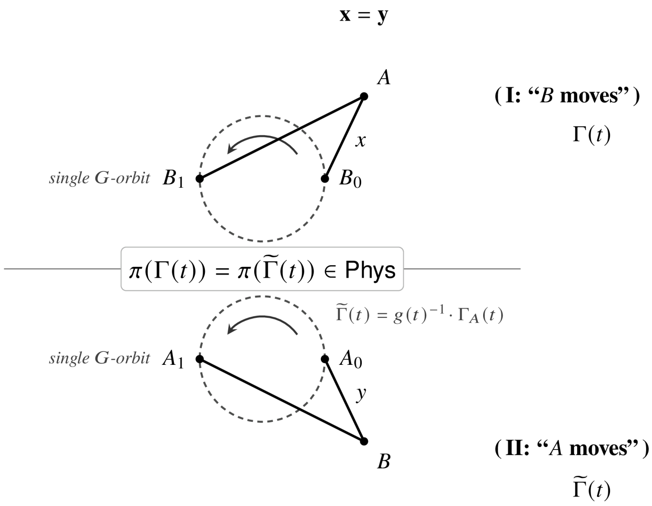

Figure 1: Relational indistinguishability of orbit-projection-equivalent descriptions. *Top half*: an A-fixed, hence B-supported, description (interpretation I: "B moves"). *Bottom half*: a B-fixed, hence A-supported, twin description (interpretation II: "A moves"). Since $\Gamma_A(t) = g(t)\cdot\widetilde{\Gamma}(t)$, one has $\pi(\Gamma(t)) = \pi(\widetilde{\Gamma}(t))$ in $\mathsf{Phys} = X/G$, and therefore $R(\Gamma(t)) = R(\widetilde{\Gamma}(t))$ for every coherent report R. The labels x and y on the comparison arrows (edges from A to B_0, B to A_0) denote comparison-relation values on specific edges of each orbit diagram; they are distinct from the notation $x = y$ at the top of the figure, which denotes equality of report values (not equality of states or comparison values). No coherent report distinguishes the two interpretations.

Proof. Since Θ_X is orbit-preserving, $\pi\circ\Theta_X = \pi$. Evaluating at x gives $\pi(\Theta_X(x)) = \pi(x)$. Hence the map induced on quotient points sends each $p = \pi(x)$ to $\pi(\Theta_X(x)) = p$, so it is $\mathrm{id}_{\mathsf{Phys}}$. □

Lemma 8.3 (Invariance under orbit-preserving maps)**.** *If Θ_X is orbit-preserving, $R\circ\Theta_X = R$ for every coherent R.*

Proof. $R = \widetilde{R}\circ\pi$, so $R(\Theta_X(x)) = \widetilde{R}(\pi(\Theta_X(x))) = \widetilde{R}(\pi(x)) = R(x)$. □

Theorem 8.4 (Quotient invisibility of orbit-preserving maps)**.** *Orbit-preserving maps act trivially on all coherent reports.*

Proof. Lemma 8.3. □

Remark 8.5 (Orbit-preserving involutions and coherent reports)**.** *Any involution preserving orbits is invisible on all coherent reports. Nontrivial structure must enter through the enrichment mechanisms of Theorem 9.10.*

8.2 Fiberwise formulation

Definition 8.6 (Fiberwise map)**.** Θ_X is *fiberwise* if for each $p\in\mathsf{Phys}$ it restricts to $\Theta_p : \pi^{-1}(p)\to\pi^{-1}(p)$.

Lemma 8.7 (Equivalence of orbit preservation and fiberwise action)**.** *Θ_X is orbit-preserving iff fiberwise, and is then uniquely determined by $\{\Theta_p\}$.*

Proof. If Θ_X is orbit-preserving and $x\in\pi^{-1}(p)$, then $\pi(\Theta_X(x)) = \pi(x) = p$, so $\Theta_X(x)\in\pi^{-1}(p)$; hence Θ_X restricts to a map $\Theta_p : \pi^{-1}(p)\to\pi^{-1}(p)$ on each fiber. Conversely, if Θ_X restricts to each fiber, then for $x\in\pi^{-1}(p)$ one has $\Theta_X(x)\in\pi^{-1}(p)$, so $\pi(\Theta_X(x)) = p = \pi(x)$. The restrictions $\{\Theta_p\}_{p\in\mathsf{Phys}}$ determine Θ_X because every $x\in X$ lies in the unique fiber $\pi^{-1}(\pi(x))$. □

Lemma 8.8 (Unique fixed points yield a section)**.** *Assume Θ_X is orbit-preserving and $\mathrm{Fix}(\Theta_X)\cap\pi^{-1}(p)$ is a singleton for each p. Then the unique fixed point defines a section $s : \mathsf{Phys}\to X$.*

Proof. For each $p\in\mathsf{Phys}$, let $s(p)$ be the unique element of $\mathrm{Fix}(\Theta_X)\cap\pi^{-1}(p)$. Then $s(p)\in\pi^{-1}(p)$, so $\pi(s(p)) = p$. Hence s is a section of π. □

Remark 8.9 (Orbit-preserving maps and representative selection)**.** *If each fiber has a unique fixed point, the orbit-preserving map determines* $s : \mathsf{Phys} \to X$*. Taking* $\Theta_X(x) = s(\pi(x))$ *gives the idempotent pinning map (*$\Theta_X^2 = \Theta_X$*); this is mechanism (i) of Theorem 9.10.*

9 Extensions of Quotient Semantics

By Theorem 8.4, any orbit-preserving map is quotient-invisible. Route dependence must arise from structure beyond orbit-level data.

9.1 Protocols and functorial extensions

Definition 9.1 (Finite protocol)**.** A *finite protocol* is $(\mathscr{N}, \gamma)$ where $\mathscr{N} = (V, E)$ is a finite directed graph and γ is a directed path. When a protocol is used as an extension of quotient semantics, its vertex set is taken to satisfy $V \subset \mathsf{Phys}$.

Definition 9.2 (Extension of quotient semantics)**.** Let $\mathrm{Path}^{\pm}(\mathscr{N})$ denote the free groupoid on a finite network $\mathscr{N} = (V, E)$ with $V \subset \mathsf{Phys}$. An *extension of quotient semantics* assigns to each such network the following data:

(i) (*Representative Locus*) optionally, a selector $s_V : V \to X$ with $\pi \circ s_V = \mathrm{id}_V$;

(ii) (*Morphism Locus*) a small comparison groupoid $\mathsf{C}_{\mathscr{N}} \rightrightarrows V$ together with an edge transport assignment sending each edge $e : p \to q$ to a morphism $\tau_e \in \mathrm{Hom}_{\mathsf{C}_{\mathscr{N}}}(p, q)$;

(iii) the unique functor $\mathscr{F}_{\mathscr{N}} : \mathrm{Path}^{\pm}(\mathscr{N}) \to \mathsf{C}_{\mathscr{N}}$ with $\mathscr{F}_{\mathscr{N}}(e) = \tau_e$ for every edge $e \in E$.

Here and below, $\mathscr{F}$ denotes the network-indexed extension assignment, while $\mathscr{F}_{\mathscr{N}}$ denotes its component functor for the particular network $\mathscr{N}$. If a selector s_V is present, it may be used in defining the transport assignment $e \mapsto \tau_e$ (so the transport values may depend on s_V), but no further auxiliary datum is allowed. When a selector is present and influences transport, the functor $\mathscr{F}_{\mathscr{N}}$ is the unique extension of the resulting selector-dependent edge assignment.

Proposition 9.3 (Determinacy by selector and transport)**.** *Within Definition 9.2, the functor* $\mathscr{F}_{\mathscr{N}}$ *is completely determined by the selector* s_V *(when present) and the morphism-level data* $(\mathsf{C}_{\mathscr{N}}, e \mapsto \tau_e)$*. Consequently the formalism admits exactly two loci of supplementary structure: object-level representative selection and morphism-level transport data.*

Proof. The object set of $\mathsf{C}_{\mathscr{N}}$ is fixed to be V, so no additional object-level datum is available beyond the selector s_V. At the morphism level, Definition 9.2(ii) supplies the comparison groupoid $\mathsf{C}_{\mathscr{N}}$ and assigns to each *generator* $e \in E$ a single morphism τ_e in that groupoid. The free path groupoid $\mathrm{Path}^{\pm}(\mathscr{N})$ is freely generated by E, so by the universal property, any such edge assignment extends to a *unique* functor $\mathscr{F}_{\mathscr{N}} : \mathrm{Path}^{\pm}(\mathscr{N}) \to \mathsf{C}_{\mathscr{N}}$. There is therefore no morphism-level datum beyond the codomain comparison groupoid together with the edge assignment. Any richer framework would add object data, generators, or morphisms not present in Definition 9.2; such additions lie outside the finitary extension formalism considered here. Within this scope, no third datum exists. The comparison groupoid itself is part of the morphism locus: it supplies the available morphisms in which the edge transports take values, rather than a separate object-level or third-locus enrichment. □

Definition 9.4 (Endpoint-determined)**.** An extension $\mathscr{F}$ is *endpoint-determined* if, for every finite network $\mathscr{N}$ and every pair of co-terminal paths $\gamma_1, \gamma_2 : p \to q$ in $\mathrm{Path}^{\pm}(\mathscr{N})$,

$$\mathscr{F}_{\mathscr{N}}(\gamma_1) = \mathscr{F}_{\mathscr{N}}(\gamma_2).$$

Remark 9.5 (Location of route dependence)**.** *By Proposition 9.3,* $\mathscr{F}_{\mathscr{N}}$ *is fixed once the selector* s_V *(if any) and the morphism-level data* $(\mathsf{C}_{\mathscr{N}}, e \mapsto \tau_e)$ *are fixed. Any failure of endpoint-determinacy is therefore witnessed by transport data on co-terminal paths, although a chosen selector may influence which transport assignment is realized.*

9.2 Endpoint-determined extensions and the two-locus criterion

Definition 9.6 (Endpoint-forced morphism)**.** $\mathscr{F}_{\mathscr{N}}(\gamma) : p \to q$ is *endpoint-forced* if every path $\gamma' : p \to q$ in $\mathrm{Path}^{\pm}(\mathscr{N})$ satisfies $\mathscr{F}_{\mathscr{N}}(\gamma') = \mathscr{F}_{\mathscr{N}}(\gamma)$.

Definition 9.7 (Loop defect in extension notation)**.** For $p \in V$, write $\Omega_p(\mathscr{N}) := \mathrm{Hom}_{\mathrm{Path}^{\pm}(\mathscr{N})}(p,p)$ and define

$$\delta_p := \mathscr{F}_{\mathscr{N}}|_{\Omega_p(\mathscr{N})} : \Omega_p(\mathscr{N}) \to \mathrm{Aut}_{\mathsf{C}_{\mathscr{N}}}(p).$$

A based loop $\ell \in \Omega_p(\mathscr{N})$ with $\delta_p(\ell) \neq \mathrm{id}_p$ is called a *loop defect*.

Lemma 9.8 (Endpoint forcing criterion)**.** *An extension $\mathscr{F}$ is endpoint-determined iff every morphism in the image of every $\mathscr{F}_{\mathscr{N}}$ is endpoint-forced.*

Proof. If $\mathscr{F}$ is endpoint-determined and $\gamma : p \to q$ is any path in $\mathrm{Path}^{\pm}(\mathscr{N})$, then every co-terminal path $\gamma' : p \to q$ satisfies $\mathscr{F}_{\mathscr{N}}(\gamma') = \mathscr{F}_{\mathscr{N}}(\gamma)$; hence $\mathscr{F}_{\mathscr{N}}(\gamma)$ is endpoint-forced. Conversely, if every realized morphism is endpoint-forced, then for any finite network $\mathscr{N}$ and any co-terminal paths $\gamma_1, \gamma_2 : p \to q$, endpoint-forcing of $\mathscr{F}_{\mathscr{N}}(\gamma_1)$ gives $\mathscr{F}_{\mathscr{N}}(\gamma_2) = \mathscr{F}_{\mathscr{N}}(\gamma_1)$. Thus $\mathscr{F}$ is endpoint-determined. □

Remark 9.9 (Structure of the exhaustion claim)**.** *Theorem 9.10 below has two components of different logical character. The first—that the supplementary data decompose into exactly two loci—is a consequence of Definition 9.2, which admits exactly two types of data (Proposition 9.3). This is a structural delimitation: in the chosen functorial extension formalism, maps from comparison networks to comparison groupoids have exactly two degrees of freedom. The second component — the equivalence* (a)⇔(b)⇔(c) *— is the genuinely nontrivial content: non-endpoint-determined behavior occurs if and only if the transport locus carries nontrivial holonomy. This characterizes* when *the two loci matter, not only* that *they exhaust the formalism.*

Theorem 9.10 (Two-locus exhaustion and holonomy characterization)**.** *Let $\mathscr{F}$ be an extension of quotient semantics (Definition 9.2). Within this framework the supplementary data of $\mathscr{F}$ decompose completely into exactly two loci:*

(i) *(*Representative Selection*) optionally, a selector $s_V : V \to X$ with $\pi \circ s_V = \mathrm{id}_V$;*

(ii) *(*Morphism Enrichment*) a comparison groupoid together with an edge transport assignment, equivalently the induced functor $\mathscr{F}_{\mathscr{N}}$ with its codomain.*

No third independent locus of additional structure is admitted by the formalism. Moreover, the following are equivalent:

(a) *$\mathscr{F}$ is not endpoint-determined;*

(b) *some realized morphism is not endpoint-forced;*

(c) *there exist a finite network $\mathscr{N} = (V,E)$, a vertex $p \in V$, and a based loop $\ell \in \Omega_p(\mathscr{N})$ with nontrivial loop defect $\delta_p(\ell) \neq \mathrm{id}_p$.*

If a selector is present, it can influence comparison data only through the transport assignment.

Proof. The decomposition into loci *(i)* and *(ii)*, and the absence of a third locus, is exactly Proposition 9.3. The equivalence *(a)⇔(b)* is Lemma 9.8. For *(b)⇒(c)*, Definition 9.2 makes $\mathscr{F}_{\mathscr{N}}$ the transport functor induced by the chosen edge assignment, so Lemma 9.30 applies and produces a based loop with nontrivial defect. For *(c)⇒(a)*, the loop $\ell : p \to p$ and the identity path at p are co-terminal but have distinct images, $\mathscr{F}_{\mathscr{N}}(\ell) = \delta_p(\ell) \neq \mathrm{id}_p = \mathscr{F}_{\mathscr{N}}(\mathrm{id}_p)$. The final sentence follows again from Proposition 9.3. □

Corollary 9.11 (Two-locus equivalence)**.** *The following are equivalent:*

(i) *$\mathscr{F}$ is not endpoint-determined.*

(ii) *There exist co-terminal paths with distinct images.*

(iii) *The morphism-level transport locus has nontrivial loop defect.*

When such non-endpoint behavior occurs, any accompanying object-level supplementary datum is necessarily representative selection via a selector s_V and, when chosen uniformly, via a section $s : \mathsf{Phys} \to X$.

Proof. (i)⇔(ii): Definition 9.4. (i)⇔(iii): Theorem 9.10. The final sentence is the object-level part of the same theorem. □

Remark 9.12 (Heuristic interpretation of the two loci)**.** *Theorem 9.10 is an* exhaustion *theorem. It identifies the types of structure that can appear and proves that no other type can appear within the present formalism.*

The two loci also admit a structural analogy. Locus (i) — representative selection via a section $s : \mathsf{Phys} \to X$ *— is analogous to gauge-fixing-like representative choice. Locus (ii) — route-dependent transport with nontrivial holonomy — is analogous to connection-like transport, where nontrivial holonomy* $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$ *for some based loop* ℓ *records the failure of path-independence. In that sense it functions as a discrete curvature-type signal. These analogies are motivational and are not used in the proofs.*

The precise content of the theorem is the internal statement that, within the finitary extension formalism of Definition 9.2, no third mechanism exists*. Any non-endpoint-determined extension must involve locus (ii), and may also use locus (i) when representative choices parameterize the transport data. Treating physical gauge theories as literal instances of this abstract framework would require additional interpretation beyond the theorem itself. What the paper proves is the structural boundary: within the present finitary formalism, the two loci are exhaustive.*

Remark 9.13 (Realization of the two enrichment mechanisms)**.** *Mechanism (i): representative selection via a selector* $s_V : V \to X$ *and, when chosen uniformly, via a section* $s : \mathsf{Phys} \to X$ *(Subsection 9.3). Mechanism (ii): transport schemes and holonomy/loop defect (Subsection 9.4). Subsection 9.3 gives a selector-dependent non-endpoint-determined realization, while Subsection 9.4 gives a pure-transport realization. The two mechanisms act at complementary categorical levels: selectors choose objects in fibers, while loop defect measures failure of endpoint-determinacy in morphisms. In Section 10, the same point appears as the elimination of a third locus in Lemma 10.7.*

Corollary 9.14 (Finitary extension criterion under closure)**.** *Assume closure in the sense of Section 4. Let* $\mathscr{F}$ *be an extension of quotient semantics (Definition 9.2). If* $\mathscr{F}$ *yields comparison distinctions not fixed by endpoint data, then its morphism-level transport datum must have nontrivial loop defect. Any additional supplementary datum present lies in one of the following forms:*

(i) *representative selection via a selector* $s_V : V \to X$ *and, when chosen uniformly, via a section* $s : \mathsf{Phys} \to X$*;*

(ii) *morphism-level enrichment, equivalently route dependence with nontrivial loop defect/holonomy.*

Without nontrivial loop defect, $\mathscr{F}$ *is endpoint-determined and introduces no invariant distinction beyond endpoint data.*

Proof. Immediate from Theorem 9.10 and Corollary 9.11. Any non-endpoint-determined extension draws from the two loci. Without nontrivial loop defect the extension is endpoint-determined. Under closure, any admissible distinction beyond the orbit-level baseline must therefore arise at one of those two loci. ☐

9.3 Representative Selection and Splitting Data

This subsection isolates mechanism (i) of Theorem 9.10: representative selection.

9.3.1 Sections

Definition 9.15 (Section)**.** A *section* of π is $s : \mathsf{Phys} \to X$ with $\pi \circ s = \mathrm{id}_{\mathsf{Phys}}$.

Remark 9.16 (Local selectors and global sections)**.** *For a finite protocol with vertex set* $V \subset \mathsf{Phys}$*, a selector* $s_V : V \to X$ *satisfying* $\pi \circ s_V = \mathrm{id}_V$ *is a partial section of* π *over* V*. Any global section* $s : \mathsf{Phys} \to X$ *induces such selectors by restriction,* $s_V = s|_V$*.*

9.3.2 Orbit-Pinning Maps

Definition 9.17 (Orbit-pinning map)**.** Θ_X is an *orbit-pinning map* if: (i) $\pi \circ \Theta_X = \pi$; (ii) $\Theta_X^2 = \Theta_X$; (iii) each fiber $\pi^{-1}(p)$ has exactly one fixed point.

Theorem 9.18 (Equivalence of sections and orbit-pinning maps)**.** π *admits a section iff there exists an orbit-pinning map.*

Proof. ($\Leftarrow$) The unique fixed point in each fiber defines $s(p)$; $\pi(s(p)) = p$. ($\Rightarrow$) Given s, define $\Theta_X(x) := s(\pi(x))$. Then $\pi \circ \Theta_X = \pi$, $\Theta_X^2 = \Theta_X$, and $s(p)$ is the unique fixed point in $\pi^{-1}(p)$. ☐

Remark 9.19 (Existence without naturality)**.** *The equivalence asserts existence only; no canonical choice exists. Representative selection is additional structure beyond quotient semantics.*

Proposition 9.20 (Selector-dependent realization)**.** *The representative locus of Definition 9.2 occurs genuinely in a non-endpoint-determined extension.*

Proof. Take a nontrivial group G and let $X := G$ with the left translation action. Then $\mathsf{Phys} = X/G$ is a singleton $\{p\}$, so a section $s : \mathsf{Phys} \to X$ is just the choice of an element $x_* := s(p) \in G$; choose $x_* \neq e_G$. Let $\mathscr{N}$ be the one-vertex network with vertex p and two loop edges e_0, e_1. Let $\mathsf{C}_{\mathscr{N}}$ be the one-object groupoid with automorphism group G, and define the transport assignment by

$$\tau_{e_0} := e_G, \qquad \tau_{e_1} := x_*.$$

The induced functor $\mathscr{F}_{\mathscr{N}} : \mathrm{Path}^{\pm}(\mathscr{N}) \to \mathsf{C}_{\mathscr{N}}$ is then non-endpoint-determined, since e_0 and e_1 are co-terminal loops at p with distinct images. Here the extra datum $x_* = s(p)$ is exactly the chosen representative. □

Remark 9.21 (How selector data enters)**.** *Proposition 9.20 does not contradict the object–morphism complementarity of Definition 9.2. The selector does not itself create new morphisms; rather, it parameterizes the transport assignment, which is where non-endpoint-determined behavior is finally witnessed.*

9.3.3 Groupoid Formulation of Splitting

Let $\mathsf{G} := G \ltimes X$ (action groupoid [9]). There is a canonical identification $\pi_0(\mathsf{G}) \cong \mathsf{Phys}$.

Theorem 9.22 (Groupoid characterization of splitting)**.** *π admits a section iff $\mathrm{pr} : \mathsf{G} \to \pi_0(\mathsf{G})$ admits a functorial section σ with $\mathrm{pr} \circ \sigma = \mathrm{id}_{\pi_0(\mathsf{G})}$.*

Proof. Given s, define $\sigma(p) := s(p)$ on objects. Since $\pi_0(\mathsf{G})$ is discrete, there are no non-identity morphisms whose images must be specified, so this object map determines a functor. Moreover $\mathrm{pr}(s(p)) = p$, hence $\mathrm{pr} \circ \sigma = \mathrm{id}_{\pi_0(\mathsf{G})}$. Conversely, a functorial section σ selects an object $\sigma(p) \in X$ in each connected component $p \in \pi_0(\mathsf{G}) \cong \mathsf{Phys}$. The assignment $s(p) := \sigma(p)$ satisfies $\pi(s(p)) = p$, and hence is a section of π. □

Proposition 9.23 (Complementarity of the two enrichment loci)**.** *Within Definition 9.2, the two enrichment loci are complementary.* (i) *A section is object-level data: by itself it creates no morphisms.* (ii) *A transport scheme is morphism-level data: by itself it changes no objects.*

Proof. (i) Follows from Theorem 9.22: $\pi_0(\mathsf{G})$ is discrete, so a section chooses objects and creates no morphisms. (ii) Transport assigns morphisms while leaving the object set V fixed. □

9.4 Transport Schemes and Holonomy

This subsection isolates mechanism (ii) of Theorem 9.10: morphism-level enrichment.

9.4.1 Networks and the Free Path Groupoid

Recall that $\mathrm{Path}^{\pm}(\mathscr{N})$ is the free groupoid on $\mathscr{N} = (V, E)$ (universal property [9]: edge assignments extend uniquely to functors).

9.4.2 Comparison Groupoid and Transport

Definition 9.24 (Comparison groupoid)**.** Given an extension as in Definition 9.2, the small groupoid $\mathsf{C}_{\mathscr{N}} \rightrightarrows V$ appearing in part *(ii)* is called its *comparison groupoid*.

Definition 9.25 (Transport scheme)**.** Given an extension as in Definition 9.2, the edge assignment specified in part *(ii)*, which sends each edge $e : p \to q$ to a morphism $\tau_e \in \mathrm{Hom}_{\mathsf{C}_{\mathscr{N}}}(p, q)$, is called its *transport scheme*.

By Definition 9.2(iii), equivalently by the universal property, this edge assignment extends uniquely to the component functor [10] $\mathscr{F}_{\mathscr{N}} : \mathrm{Path}^{\pm}(\mathscr{N}) \to \mathsf{C}_{\mathscr{N}}$. In the transport and loop-defect discussion it is useful to write this same functor in holonomy notation:

$$\mathrm{Hol} := \mathscr{F}_{\mathscr{N}} : \mathrm{Path}^{\pm}(\mathscr{N}) \to \mathsf{C}_{\mathscr{N}}, \tag{9.1}$$

with $\mathrm{Hol}(e) = \tau_e$ and $\mathrm{Hol}(\gamma) = \tau_{e_{n-1}} \circ \cdots \circ \tau_{e_0}$. No additional datum is introduced by the notation Hol.

9.4.3 Route Dependence and Loop Defect

Definition 9.26 (Endpoint-determined and route-dependent transport)**.** A transport scheme is *endpoint-determined* if co-terminal paths always have equal Hol-images; otherwise *route-dependent*.

Lemma 9.27 (Loop reduction)**.** *For co-terminal paths $\gamma_1, \gamma_2 : p \to q$,* $\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2) \iff \mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{id}_p$.

Proof. $\mathrm{Hol}(\gamma_2^{-1} \circ \gamma_1) = \mathrm{Hol}(\gamma_2)^{-1} \circ \mathrm{Hol}(\gamma_1)$ by functoriality of Hol. This composite is id_p if and only if $\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2)$. □

Theorem 9.28 (Equivalence of route dependence and loop defect)**.** *A transport scheme is route-dependent if and only if there exists a based loop $\ell : p \to p$ with* $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$.

Proof. If route-dependent, take $\ell := \gamma_2^{-1} \circ \gamma_1$ for paths with $\mathrm{Hol}(\gamma_1) \neq \mathrm{Hol}(\gamma_2)$; Lemma 9.27 gives $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$. Conversely, if all loop images are trivial, Lemma 9.27 forces all co-terminal paths to have equal Hol-images. □

Lemma 9.29 (Path ambiguity implies loop defect)**.** *Let $\gamma_1, \gamma_2 : p \to q$ be co-terminal paths with* $\mathrm{Hol}(\gamma_1) \neq \mathrm{Hol}(\gamma_2)$. *Then $\ell := \gamma_2^{-1} \circ \gamma_1$ lies in $\Omega_p(\mathscr{N})$ and satisfies $\delta_p(\ell) = \mathrm{Hol}(\ell) \neq \mathrm{id}_p$.*

Proof. By Lemma 9.27, $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$. ℓ is a based loop at p, so $\ell \in \Omega_p(\mathscr{N})$. $\delta_p(\ell) = \mathrm{Hol}(\ell)$ by definition of δ_p. □

Lemma 9.30 (Formal transport implication in extension notation)**.** *If $\mathscr{F}_{\mathscr{N}}(\gamma_1) \neq \mathscr{F}_{\mathscr{N}}(\gamma_2)$ for co-terminal paths $\gamma_1, \gamma_2 : p \to q$, then there exists $\ell \in \Omega_p(\mathscr{N})$ with $\delta_p(\ell) \neq \mathrm{id}_p$.*

Proof. By Definition 9.2, the edge assignment of the extension is a transport scheme and its induced transport functor is the functor Hol defined in (9.1). Take $\ell := \gamma_2^{-1} \circ \gamma_1$ and apply Lemma 9.29. □

Proposition 9.31 (Pure transport realization)**.** *The morphism locus of Definition 9.2 is realizable without any representative selector.*

Proof. Let $\mathscr{N}$ be the one-vertex network with vertex p and two loop edges e_0, e_1. Let $\mathsf{C}_{\mathscr{N}}$ be the one-object groupoid with automorphism group $\mathbb{Z}/2 = \{0,1\}$ written additively, and choose no selector. Define the transport assignment by

$$\tau_{e_0} := 0, \qquad \tau_{e_1} := 1.$$

Then $\mathrm{Hol}(e_0) \neq \mathrm{Hol}(e_1)$, so the transport scheme is route-dependent. Equivalently, the based loop $e_1 \circ e_0^{-1}$ has nontrivial defect. Hence this gives a non-endpoint-determined extension supported purely at the morphism locus. □

9.4.4 Triangle Coherence in the Single-Object Case

With $\mathsf{C}_{\mathscr{N}}$ single-object (automorphism group K), transport reduces to $\tau : E \to K$, extending to $\widehat{\tau} : \mathbb{F}(E) \to K$, where $\mathbb{F}(E)$ denotes the free group on the edge set E. In this single-object case, the identity morphism id_p coincides with the group identity $e_K \in K$; both notations are used below, id_p in groupoid statements and e_K in group-theoretic ones.

Lemma 9.32 (Triangle coherence condition)**.** *The following are equivalent: (i) $\widehat{\tau}$ is trivial on the normal closure $\mathscr{R}$ of triangle words $w_\ell = [e_{rp}][e_{qr}][e_{pq}]$; (ii) $\tau_{e_{rp}}\tau_{e_{qr}}\tau_{e_{pq}} = e_K$ for every directed triangle (the operationally useful form); (iii) $\widehat{\tau}$ factors through $\mathbb{F}(E)/\mathscr{R}$. These equivalences follow directly from the definitions of $\mathscr{R}$ and quotient groups; the geometric content is that condition (ii) is the discrete flatness condition, which Remark 9.33 interprets as the finitary analogue of a vanishing curvature form.*

Proof. $\mathscr{R} \subseteq \ker(\widehat{\tau})$ iff each triangle word maps to e_K (giving (i)⇔(ii)). Equivalence with (iii) is the universal property of quotient groups applied to $\mathbb{F}(E)$. □

Remark 9.33 (Triangle coherence and loop defect)**.** *Lemma 9.32 gives the finitary form of the statement that path-independent transport is characterized by trivial transport around loops. The triangle coherence condition $\tau_{e_{rp}}\tau_{e_{qr}}\tau_{e_{pq}} = e_K$ is the discrete analogue of a local flatness condition. When it fails, the transport scheme has a nontrivial loop defect* $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$ *for some based loop ℓ at a vertex p. This is the finitary analogue of curvature as failure of path-independence; no differential-geometric structure is assumed here.*

9.4.5 Admissible Defect Reports

Fix $p \in V$. Set $K_p := \mathrm{Aut}_{\mathsf{C}_{\mathscr{N}}}(p)$, and recall $\Omega_p(\mathscr{N})$ and $\delta_p : \Omega_p(\mathscr{N}) \to K_p$ from Definition 9.7. Let $q_p : K_p \to K_p/\sim$ be conjugacy-class projection, where $\sim$ denotes conjugacy in K_p.

Definition 9.34 (Admissible defect report)**.** $h : \Omega_p(\mathscr{N}) \to S$ is *admissible* if there exists $\widetilde{h} : K_p/\sim \to S$ with $h = \widetilde{h} \circ q_p \circ \delta_p$.

Lemma 9.35 (Conjugacy invariance)**.** *If h is admissible and $u, \ell \in \Omega_p(\mathscr{N})$, then $h(u\ell u^{-1}) = h(\ell)$.*

Proof. $\delta_p(u\ell u^{-1}) = \delta_p(u)\delta_p(\ell)\delta_p(u)^{-1}$ is conjugate to $\delta_p(\ell)$; q_p eliminates conjugacy dependence. □

Remark 9.36 (Elements and conjugacy classes)**.** *The conjugacy class $q_p(\delta_p(\ell))$ is canonical. Selecting a specific element requires additional structure beyond quotient semantics.*

10 Conditional Closure Rigidity and Openness Witnesses

This section recasts universe-level closure as a witness-elimination problem. Within the present framework, failure of closure or operationally meaningful structure beyond the closure conditions adopted here can appear only through one of the witness types listed below. Each listed type is then excluded by results already proved once the corresponding closure hypotheses are imposed.

Definition 10.1 (Witness of openness)**.** A *witness of openness* is one of the following.

(W0) *Profile-Map Defect Witness:* the canonical factor map $\Theta : U \to X_A \times X_B$ is not bijective. Equivalently, either distinct states have the same left-right profile pair, or some left-right profile pair is not realized by any state of U.

(W1) *External-Reference Witness:* there exist $L \neq \varnothing$ and a physically interpreted map $F : L \times X \to S$ whose dependence on L is not eliminated by $\mathrm{pr}_X : L \times X \to X$.

(W2) *Internal-Admissibility Witness:* there exists a purportedly admissible observable on X that is not internally generated (Definition 4.1), or is internally generated but not protocol-local (Definition 4.2).

(W3) *Third-Locus Witness:* there exists a non-endpoint-determined extension of quotient semantics (Definition 9.2) whose additional distinguishability is realized by neither representative selection nor morphism-level enrichment.

Remark 10.2 (Organization of witness types)**.** *Definition 10.1 introduces no new axiom. It serves only to organize the possibilities recognized by the present framework: profile-map defects, external reference data, failure of internal admissibility, or a third enrichment locus beyond representative selection and transport.*

Lemma 10.3 (Elimination of profile-map defect witnesses)**.** *If $(U, \mathscr{C})$ is rectangularly complete, then no witness of type* (W0) *exists.*

Proof. By Theorem 2.10, rectangular completeness is equivalent to bijectivity of $\Theta : U \to X_A \times X_B$. Hence the canonical profile map has neither collisions nor missing profile pairs, so no profile-map defect witness exists. □

Lemma 10.4 (Elimination of external witnesses)**.** *Under external-label symmetry, no witness of type* (W1) *exists.*

Proof. For witnesses of type *(W1)* satisfying the external-label symmetry condition, Theorem 1.1 says that any $F : L \times X \to S$ satisfying (1.3) factors through pr_X, and through $\pi \circ \mathrm{pr}_X$ when slices are coherent. Hence external-label dependence contributes no additional invariant content. □

Lemma 10.5 (Elimination of internal-admissibility witnesses)**.** *Assume operational closure (Definition 4.4) and the closed-world convention that admissible observables are internally generated. Then no witness of type* (W2) *exists.*

Proof. Operational closure gives protocol locality for internally generated reports. By Theorem 4.10, such reports factor through $\pi : X \to \mathsf{Phys}$. Under the stated convention, admissible observables are precisely the internally generated reports covered by operational closure; hence type *(W2)* is excluded. □

Remark 10.6 (Scope of the W2 elimination)**.** *The W2 elimination identifies the scope of "admissible observable" within the closed-system framework: operational closure forces factorization through* π *(Theorem 4.10), and observables not of this class lie outside the closed-world admissibility criterion adopted here. Thus the framework draws its boundary at protocol-local internally generated reports, and within that boundary all structure is accounted for by the descent and two-locus results.*

Lemma 10.7 (Elimination of third-locus witnesses)**.** *Within the finitary framework of Definition 9.2, no witness of type* (W3) *exists.*

Proof. By Theorem 9.10 and Corollary 9.11, every non-endpoint-determined extension draws its additional distinguishability from the object-level or morphism-level locus isolated there. Hence no third independent locus is encoded by the formalism. □

Theorem 10.8 (Conditional universe-level closure rigidity)**.** *Assume:*

(i) *the modeled system is the universe;*

(ii) *the underlying comparison world is rectangularly complete;*

(iii) *external-label symmetry* (1.3) *holds for putative external labels;*

(iv) *operational closure holds for internally generated reports;*

(v) *admissible state observables are internally generated;*

(vi) *extensions are in the finitary framework of Definition 9.2.*

Then no openness witness of the types in Definition 10.1 exists. In this terminology:

(a) *every admissible state report factors through* $\pi : X \to \mathsf{Phys}$*;*

(b) *every admissible non-endpoint comparison distinction is realized only at the two loci isolated in Theorem 9.10.*

Proof. Within the taxonomy of Definition 10.1, openness is represented by at least one witness type *(W0)–(W3)*. Lemmas 10.3, 10.4, 10.5, and 10.7 eliminate all four. Hence no such openness witness exists. Part *(a)* follows from Theorem 4.10. Part *(b)* follows from Theorem 9.10 and Corollary 9.11. □

Remark 10.9 (Interpretation of universe-level closure rigidity)**.** *Theorem 10.8 is conditional on its listed hypotheses. In particular, rectangular completeness, external-label symmetry, operational closure, internal generation of admissible observables, and the finitary extension formalism are not derived from first principles here; they are the formal expression of adopting the closed-system framework studied in this paper. The theorem's content is therefore not an absolute impossibility result, but a rigidity theorem within the adopted framework: given these hypotheses, the only structures that can appear within the stated witness taxonomy are those already isolated by the preceding results. This is the appropriate reading of the no-openness-witness conclusion.*

Remark 10.10 (Degenerate case: trivial symmetry group)**.** *If* $G = \{e_G\}$ *is trivial, the orbit projection is* $\pi = \mathrm{id}_X$ *and* $\mathsf{Phys} = X$*. Every map* $R : X \to S$ *is coherent, the single-orbit property is trivial (pure orbit loops are constant), and the subsystem-attribution obstruction is vacuous. In the two-locus framework, extensions of the identity quotient are arbitrary transport schemes on networks with vertex set* X*, and all theorems apply (endpoint-determinacy becomes the condition that transport is path-independent on* X*). This degenerate case is consistent with all results and illustrates that the nontrivial content of the paper arises precisely when* G *is nontrivial and generates genuinely non-singleton orbits.*

11 Discussion

11.1 The logical chain

The argument proceeds as a chain of theorem-level implications once the closure and admissibility criteria have been fixed. Starting from a comparison world $(U, \mathscr{C})$—a set of states with binary predicates and nothing else—rectangular completeness (Definition 2.6) is the minimal internal condition for closure

(Theorem 2.10). By Theorem 2.19, within locally distinguishable worlds the same condition is equivalent to profile-maximality: the absence of any proper locally distinguishable profile-determined one-point extension. Equivalently,

$$\text{closure} \iff \text{profile-maximality against proper profile-determined one-point extensions.}$$

From this starting point one obtains:

(1) a canonical product decomposition $U \cong X_A \times X_B$, with $X_A = U/\alpha$ and $X_B = U/\beta$ the quotients by the intrinsic left- and right-profile congruences (Theorem 2.8);

(2) a diagonal action of the intrinsic symmetry group $G = \mathrm{Aut}(U, \mathscr{C})$ on $X = X_A \times X_B$ (Theorem 2.8(ii));

(3) the orbit projection $\pi : X \to \mathsf{Phys} := X/G$, canonically determined (Theorem 2.8(iii));

(4) forced quotient semantics: by Theorems 3.2 and 4.6, every admissible closed-system report factors uniquely through π;

(5) the obstruction to subsystem attribution: no coherent report can identify which subsystem moved during pure orbit motion, and no orbit-projection-invariant functional can consistently attribute support (Theorem 7.1, Theorem 7.3);

(6) the two-locus exhaustion: within the finitary extension formalism, no third mechanism for introducing structure beyond quotient-level content exists (Theorem 9.10).

Every step in this chain is proved. The group action is derived, not postulated; quotient semantics is obtained from the closed-system admissibility criterion, equivalently from operational closure for internally generated reports; and the Two-Locus Theorem is exhaustive: it identifies all enrichment loci allowed by the formalism. The preliminary maximality reformulation shows that even the closure premise itself is rigid at the level of comparison worlds. In particular, rectangular completeness is the minimal relational condition that forces a canonical two-subsystem decomposition.

11.2 Physical content of the obstruction

The no-attribution result concerns what closed comparison data determine. In a two-body description, a changing separation, angle, impact parameter, transition relation, or outcome correlation can be a perfectly admissible report whenever it is invariant under the diagonal redundancy. Quotient data leave open the support attribution of a pure orbit change: whether the same quotient history is represented as motion of A, motion of B, or a common change of representative.

This is why the theorem targets subsystem attribution rather than motion itself. The closed comparison world determines the orbit history in Phys. It does not determine a preferred lift of that history to $X_A \times X_B$. A preferred lift is exactly additional representative-selection structure. Likewise, if a comparison protocol distinguishes two routes with the same endpoints, the distinguishing datum is not endpoint content in Phys; it is transport with nontrivial loop defect. Thus the physical alternatives are concrete: choose representatives, introduce route-dependent transport, or remain at quotient-level content. This is the content of the two-locus theorem.

11.3 What the exhaustion theorem establishes

Theorem 9.10 is the paper's central structural result. It identifies the exact ways quotient-level state content can be supplemented inside Definition 9.2: object-level representative selection and morphism-level route-dependent transport. The first chooses lifts in the fibers of π; the second distinguishes coterminal paths by nontrivial holonomy, equivalently by a loop defect $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$.

This is a structural boundary theorem. Interpreting it as a statement about a particular physical gauge theory would require additional physical input beyond the finitary comparison formalism used here.

11.4 Descent and realization

The two-locus result has both necessity and realization directions. The necessity direction says that any non-endpoint-determined extension of quotient semantics draws supplementary structure from one or both of exactly two loci. The realization direction adds sufficiency: nontrivial transport, with or without selector-dependence, produces non-endpoint-determined behavior.

Corollary 11.1 (Necessity and sufficiency of the two loci)**.** *Let $\mathscr{F}$ be a functorial extension of quotient semantics.*

(i) *(*Necessity.*) If $\mathscr{F}$ is non-endpoint-determined, then its morphism-level transport datum carries nontrivial holonomy: there exists a based loop $\ell : p \to p$ with* $\mathrm{Hol}(\ell) \neq \mathrm{id}_p$. *Any accompanying object-level datum is necessarily representative selection at locus (i) of Theorem 9.10.*

(ii) *(*Sufficiency.*) Nontrivial holonomy at the morphism locus alone suffices to make $\mathscr{F}$ non-endpoint-determined (Proposition 9.31). A selector at locus (i), together with a transport assignment that it parameterizes and that produces nontrivial holonomy, likewise yields a non-endpoint-determined extension (Proposition 9.20). A selector present without any nontrivial holonomy does* not *by itself force non-endpoint-determinacy.*

Proof. Part *(i)* is the necessity direction of Theorem 9.10 together with Proposition 9.3. For part *(ii)*, the pure-transport case is Proposition 9.31, and the selector-parameterized case is Proposition 9.20. The final sentence of part *(ii)* follows by contrapositive from Theorem 9.10*(a)⇔(c)*. If every based loop has trivial defect, then $\mathscr{F}$ is endpoint-determined regardless of whether a selector is present. □

The corollary cleanly separates necessity from sufficiency. Necessity requires nontrivial holonomy at the morphism locus; the object locus (selector) may co-occur, but it cannot substitute for nontrivial transport. Sufficiency is sharper: nontrivial holonomy alone suffices, while a selector enriches the object level without by itself producing non-endpoint-determined behavior.

A map that factors through π — that is, a map determined by Phys alone — cannot distinguish states within a single orbit. Supplying a section $s : \mathsf{Phys} \to X$ is therefore the minimal object-level datum that selects an intra-orbit representative, and it may parameterize a transport assignment with nontrivial holonomy. Supplying a transport scheme with nontrivial holonomy is the minimal morphism-level datum that distinguishes co-terminal paths beyond their common endpoints, and it is the necessary and sufficient condition for non-endpoint-determined behavior.

11.5 Scope and limitations

The analysis is restricted to the two-subsystem case and to finitary functorial extensions. The two-subsystem restriction is partly technical and partly conceptual: pairwise comparison is the basic unit in the present framework, whereas three or more subsystems introduce additional overlap constraints between distinct pairings. The pairwise setting is treated under the stated hypotheses.

For three or more subsystems, those overlap constraints impose additional coherence conditions on the transport data and point again toward the loop defects isolated above. A systematic treatment is not part of the present analysis. The same is true for continuous curvature, Hilbert-space structure, field-theoretic examples, operator-algebraic settings, and higher-categorical extensions.

The restriction also has a structural motivation. The present framework isolates the pairwise relational core from which richer comparison schemes are built. Once one moves beyond a single pair to overlapping configurations, additional coherence data become unavoidable. Familiar examples include transport data, with holonomy measuring failure of path-independence, and representative-selection data, abstracted here by sections. From that perspective, the two loci identified above serve as abstract counterparts of two recurrent moves in physical theory: transport-type enrichment and representative selection.

12 Conclusion

The paper identifies the obstruction boundary for subsystem attribution in a closed two-subsystem world. Within the present formalism, closed-world admissibility requires reports to be recoverable from the system's own comparison data rather than from external scaffolding. That requirement forces quotient semantics, rules out invariant attribution along pure orbit motion, and admits further structure at exactly two loci. These conclusions require no gauge choices, topology, or dynamics. At the level of comparison worlds themselves, within the locally distinguishable case and for intrinsic one-point adjunctions, closed and maximal are the same condition.

More specifically, once rectangular completeness is imposed as the minimal internal closure condition — equivalently, within locally distinguishable comparison worlds, once closure is formulated as profile-maximality against proper profile-determined one-point extensions — the subsystem decomposition, the diagonal symmetry action, and the orbit quotient are derived rather than assumed. Admissible reports therefore descend to $\mathsf{Phys} = X/G$, and pure orbit motion admits no invariant rule for saying which subsystem moved. When the extracted group G is trivial, this obstruction degenerates to the vacuous case $\mathsf{Phys} = X$; the substantive case is the one in which the comparison data leave nontrivial common transformations as automorphisms. Any enrichment beyond quotient-level content must come from representative selection, route-dependent transport with nontrivial holonomy, or both.

These results set a structural baseline for later physical interpretation and for extensions to settings with more than two subsystems, continuous structure, or higher-categorical semantics. They also complement relational approaches to observables in closed systems [1, 3].

A Reparametrization and Forced Descent

Fix $T > 0$, Y a set, and $\mathscr{H}$ a set of histories $\Gamma : [0,T] \to Y$. Assume $\mathscr{H}$ is closed under the reparametrizations considered below. Let $\mathscr{G} \le \mathrm{Bij}([0,T])$ act by $(\phi \cdot \Gamma)(t) := \Gamma(\phi^{-1}(t))$.

A.1 Induced group action on histories

Lemma A.1 (Left group action on histories). *The assignment $(\phi \cdot \Gamma)(t) := \Gamma(\phi^{-1}(t))$ defines a left action.*

Proof. For closure, ϕ^{-1} is a bijection of $[0,T]$, so $\phi \cdot \Gamma$ is again a history in the chosen reparametrization class. For the identity bijection $\mathrm{id}_{[0,T]}$, $(\mathrm{id}_{[0,T]} \cdot \Gamma)(t) = \Gamma(t)$. For $\phi, \psi \in \mathscr{G}$,

$$(\phi \cdot (\psi \cdot \Gamma))(t) = (\psi \cdot \Gamma)(\phi^{-1}(t)) = \Gamma(\psi^{-1}(\phi^{-1}(t))) = \Gamma((\phi\psi)^{-1}(t)) = ((\phi\psi) \cdot \Gamma)(t),$$

where $(\phi\psi)(s) = \phi(\psi(s))$. Hence the assignment is a left action. □

A.2 Orbit quotient

Define $\mathsf{Phys}_{\mathscr{H}} := \mathscr{H}/\mathscr{G}$ with projection $\pi_{\mathscr{H}} : \mathscr{H} \to \mathsf{Phys}_{\mathscr{H}}$.

A.3 Descent under reparametrization

Closed-History Hypothesis. Admissible reports are $\mathscr{G}$-invariant maps $\mathscr{O} : \mathscr{H} \to \mathbb{R}$.

Theorem A.2 (Forced descent for reparametrization orbits). *$\mathscr{O}$ is $\mathscr{G}$-invariant iff it is constant on orbits iff $\mathscr{O} = \widetilde{\mathscr{O}} \circ \pi_{\mathscr{H}}$ for a unique $\widetilde{\mathscr{O}}$.*

Proof. Apply Theorem 3.2 to the action $\mathscr{G} \curvearrowright \mathscr{H}$. □

Corollary A.3 (Orbit indistinguishability). $\Gamma' = \phi \cdot \Gamma \Rightarrow \mathscr{O}(\Gamma') = \mathscr{O}(\Gamma)$.

Proof. If $\Gamma' = \phi \cdot \Gamma$, then Γ' and Γ lie in the same $\mathscr{G}$-orbit. The claim follows from Theorem A.2. □

This recasts reparametrization invariance as a forced-descent statement and requires no Hamiltonian assumptions. The appendix adds no new mathematics beyond Theorem 3.2 applied to the reparametrization action; it is included to illustrate that the descent mechanism is not specific to the spatial-subsystem setting of the main text.

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Author Contributions

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